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Existence of Solutions of BVPs for Nonlinear Difference Equations: Non-resonance Case[1]

Yuji Liu

Department of Mathematics, Hunan Institute of Science and Technology, Yueyang 414000, P.R.China
e-mail: liuyuji888@sohu.com

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Abstract. Sufficient conditions for the existence of solutions of the following boundary value problem for nonlinear difference equation

$$\begin{cases} x(t+2k) - \sum_{i=0}^{2k-1} a_i x(t+i) = f(t, x(t), x(t+1), \cdots, x(t+2k-1)), \\ \hfill t \in [0, T-1], \\ x(i) = A_i, \;\; i \in [0, k-1], \\ x(T+2k-1-i) = B_i, \;\; i \in [0, k-1] \end{cases}$$

at non-resonance case are established. Examples are given to show the efficiency of the result in this paper.

Key words: Solution; nonlinear difference equation; boundary value problem; fixed-point theorem; growth condition.
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1. Introduction

Recently there has been a large amount of attention paid to the existence of solutions of boundary value problems for the differential equations that arise from various applied problems. Similarly there has been a parallel interest in results for the analogous discrete problems, see the text books [1,2], the papers [3-9,11-14] and the references therein.
Particular significance in these points lies in the fact that when a BVP is discretized, strange and interesting changes can occur in the solutions. For example, properties such as existence, uniqueness and multiplicity of solutions may

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not be shared between the continuous differential equation and its related discrete difference equation [3, p. 520]. Moreover, when investigating difference equations, as opposed to differential equations, basic ideas from calculus are not necessarily available to use, such as the intermediate value theorem, the mean value theorem and Rolle′s theorem. Thus, new challenges are faced and innovation is required [6].
In [4], the following boundary value problem (BVP) involving second order difference equations and two-point boundary conditions

$$(1) \qquad \begin{cases} \frac{\nabla\triangle y_k}{h^2} = f\left(t_k, y_k, \frac{\triangle y_k}{h}\right), \ k = 1, \cdots, n-1, \\ y_0 = A, y_n = B, \end{cases}$$

was studied, where $A, B \in R$, $n \geq 2$ an integer, f is continuous, scalar-valued function, the step size is $h = N/n$ with N a positive constant, the grid points are $t_k = kh$ for $k = 0, \cdots, n$. The differences are given by

$$\triangle y_k = \begin{cases} y_{k+1} - y_k, & k = 0, \cdots, n-1, \\ 0, & k = n; \end{cases}$$

$$\nabla \triangle y_k = \begin{cases} y_{k+1} - 2y_k + y_{k-1}, & k = 1, \cdots, n-1, \\ 0, & k = 0 \ \text{ or } \ k = n. \end{cases}$$

The following two results were proved in [4].

Theorem RT1. Let f be continuous on $[0, n] \times R^2$ and α, β and K be non-negative constants. If there exist $c, d \in [0, 1)$ such that

$$(2) \qquad |f(t, u, v)| \leq \alpha|u|^c + \beta|v|^d + K, (t, u, v) \in [0, n] \times R^2,$$

then the discrete BVP(1) has at least one solution.

Theorem RT2. Let f be continuous on $[0, n] \times R^2$ and α, β and K be non-negative constants. If

$$(3) \qquad |f(t, u, v)| \leq \alpha|u| + \beta|v| + K, (t, u, v) \in [0, n] \times R^2,$$

and

$$(4) \qquad \frac{\alpha n^2}{8} + \frac{\beta n}{2} < 1,$$

then the discrete BVP(1) has at least one solution.

In Theorems RT1 and RT2, the assumptions (2) or (3) allow f to grow either sublinearly or at most linearly. The problem appears naturally:

Problem 1. Under what conditions does BVP(1) has at lest one solution when f grows superlinearly?

To solve problem 1, in [6], the authors investigated the solvability of a class of discrete Dirichlet boundary value problems

$$\begin{cases} \frac{\nabla\triangle y_k}{h^2} = f\left(t_k, y_k, \frac{\triangle y_k}{h}\right), \ k = 1, \cdots, n-1, \\ y_0 = 0, y_n = 0, \end{cases}$$

by using the lower and upper solution method, where the nonlinear term $f(t,u,v)$ can have a superlinear growth both in u and in v. Moreover, the growth conditions on f are one-sided. By computing the priori bounds on solutions to the discrete problem and then obtaining the existence of at least one solution, it is shown that solutions of the discrete problem will converge to solutions of the corresponding ordinary differential equations.
In paper [5,7], the authors studied the following boundary value problems

$$\begin{cases} \Delta^2 y_{k+1} = f(t_k, y_k, \Delta y_k), \ k = 1, \cdots, n-1, \\ y_0 = A, y_n = B, \end{cases}$$

and

$$\begin{cases} \frac{\Delta^2 y_{k+1}}{h^2} = B(t_k)y_k + \frac{F(t_k)\Delta y_k}{h} + g(t_k), \ k = 1, \cdots, n-1, \\ y_0 = A, y_n = B, \end{cases}$$

respectively, where $A, B \in R^d$, $f \in C(\{0,1,\cdots,n\} \times R^{2d} \times R^d)$,

$$\triangle y_k = \begin{cases} y_{k+1} - y_k, & k = 0, \cdots, n-1, \\ 0, & k = n; \end{cases}$$

$$\nabla \triangle y_k = \begin{cases} y_{k+1} - 2y_k + y_{k-1}, & k = 1, \cdots, n-1, \\ 0, & k = 0 \ \text{ or } k = n. \end{cases}$$

The following theorem was proved.

Theorem TT [5]. Let α, K, R be nonnegative constants satisfying $2\alpha R < 1$ and set $\beta = \max\{||A||, ||B||\}$. Suppose that f satisfies

$$||f(x,y,p)|| \le \alpha(< y, f(x,y,p) > + ||p||^2) + K$$

for all $x \in \{0,1,\cdots,n\}, ||y|| \le R, p \in R^d$ with $< y, f(x,y,p) > \ge 0$. If $\beta + \alpha\beta^2 + \frac{Kn}{4} \le R$, then above problem has at least one solution.

In this paper, we study the following boundary value problem for the higher order nonlinear difference equation

$$(5) \quad \begin{cases} x(t+2k) - \sum_{i=0}^{2k-1} a_i x(t+i) = f(t, x(t), x(t+1), \cdots, x(t+2k-1)), \\ x(i) = A_i, \ \ i \in [0, k-1], \\ x(T+2k-1-i) = B_i, \ \ i \in [0, k-1], \end{cases}$$

where k, m are positive integers, $a_i(i \in [0, 2k-1] \in R$, $A_i, B_i \in R$, $T \ge 2k$ an integer, $[a,b] = \{a, a+1, \cdots, b\}$ for nonnegative integers a, b with $a \le b$, f is continuous, scalar-valued function.

It is easy to see that the discretized form of BVP of the continuous type

$$\text{(A)} \qquad \begin{cases} x''(t) = g(t, x(t), x'(t)), & t \in (0,1), \\ x(0) = A_0, \ \ x(1) = B_0 \end{cases}$$

is as follows
(B)

$$\begin{cases} x(t+2) - 2x(t+1) + x(t) = g(t, x(t), x(t+1) - x(t)), & t \in [0, T-1], \\ x(0) = A_0, \ \ x(T+1) = B_0. \end{cases}$$

One can see that BVP(B) is a spacial case of BVP(5) by choosing $k = 1$, $a_1 = -2, a_0 = 1$ and replacing $g(t, x(t), x(t+1) - x(t))$ by $f(t, x(t), x(t+1))$ in (5).
Consider the the corresponding homogenous BVP of BVP(5)

$$\text{(6)} \qquad \begin{cases} x(t+2k) - \sum_{i=0}^{2k-1} a_i x(t+i) = 0 \ \ t \in [0, T-1] \\ x(i) = 0, \ \ i \in [0, k-1], \\ x(T+2k-1-i) = 0, \ \ i \in [0, k-1]. \end{cases}$$

We call BVP(5) non-resonance if BVP(6) has unique solution $x(n) = 0$ for all $n \in [0, 2k+T-1]$ and resonance case if BVP(6) has infinitely many solutions. The purpose of this paper is to establish new sufficient conditions guaranteeing the existence of solutions of BVP (5) at non-resonance case. Problem 1 is solved by the way different from those used in known related papers [4-7]. The methods used in this paper to get the priori bound are different from those in [4-7].
This paper is organized as follows. In Section 2, the existence results of BVP(5) at non-resonance case are given. We give examples to illustrate the main result in Section 3.

2. Solvability at Non-resonance Case

Define the matrix sequence C_n by

$$\begin{aligned}
C_0 &= (a_k, a_{k+1}, \cdots, a_{2k-1}), \\
C_1 &= a_{2k-1}C_0 + (a_{k-1}, a_k, \cdots, a_{2k-2}), \\
C_2 &= a_{2k-1}C_1 + a_{2k-2}C_0 + (a_{k-2}, a_{k-1}, \cdots, a_{2k-3}), \\
C_3 &= a_{2k-1}C_2 + a_{2k-2}C_1 + a_{2k-3}C_0 + (a_{k-3}, a_{k-2}, \cdots, a_{2k-4}), \\
&\quad \cdots\cdots \\
C_k &= a_{2k-1}C_{k-1} + a_{2k-2}C_{k-2} + \cdots + a_kC_0 + (a_0, a_1, \cdots, a_{k-1}), \\
C_{k+1} &= a_{2k-1}C_k + a_{2k-2}C_{k-1} + \cdots + a_{k-1}C_0 + (0, a_0, \cdots, a_{k-2}), \\
C_{k+2} &= a_{2k-1}C_{k+1} + a_{2k-2}C_k + \cdots + a_{k-2}C_0 + (0, 0, a_0, \cdots, a_{k-3}), \\
&\quad \cdots\cdots \\
C_{2k-1} &= a_{2k-1}C_{2k-2} + \cdots + a_1C_0 + (0, \cdots, 0, a_0), \\
C_{2k} &= a_{2k-1}C_{2k-1} + a_{2k-2}C_{2k-2} + \cdots + a_0C_0,
\end{aligned}$$

$$\begin{aligned} C_{2k+1} &= a_{2k-1}C_{2k} + a_{2k-2}C_{2k-1} + \cdots + a_0C_1, \\ &\cdots\cdots \\ C_{T-1} &= a_{2k-1}C_{T-2} + a_{2k-2}C_{T-3} + \cdots + a_0C_{T-2k-1}. \end{aligned}$$

Lemma 2.1. Let $R(C)$ denote the rank of the matrix

$$C = \begin{pmatrix} C_{T-k} \\ C_{T-k+1} \\ \cdots \\ C_{T-1} \end{pmatrix}_{k\times k}.$$

Then BVP(6) has unique solution $x(n) = 0$ for all $n \in [0, T+2k-1]$ if and only if $R(C) = k$.

Proof. It follows from (6) that

$$\begin{aligned} x(2k) &= \sum_{i=0}^{2k-1} a_i x(i) = \sum_{i=k}^{2k-1} a_i x(i) = C_0 \begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix}, \\ x(2k+1) &= \sum_{i=0}^{2k-1} a_i x(1+i) = (a_{2k-1}C_0 + (a_{k-1}, \cdots, a_{2k-2})) \begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix} \\ &= C_1 \begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} x(2k+2) &= C_2 \begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix}, \\ x(2k+3) &= C_3 \begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix}, \\ &\cdots\cdots \\ x(3k) &= C_k \begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix}, \\ &\cdots\cdots \end{aligned}$$

$$
\begin{aligned}
x(k+T-1) &= C_{T-k-1}\begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix}, \\
x(k+T) &= C_{T-k}\begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix}, \\
&\cdots\cdots \\
x(2k+T-1) &= C_{T-1}\begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix}.
\end{aligned}
$$

Since $x(T+2k-1-i) = 0$ for all $i \in [0, k-1]$, we get that

$$
C \times \begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix} = 0. \tag{7}
$$

One can finds easily that $x(k) = x(k+1) = \cdots = x(2k+T-1) = 0$ if and only if $R(C) = k$. Thus BVP(6) has unique solution $x(n) = 0$ for all $n \in [0, T+2k-1]$ if and only if $R(C) = k$. The proof is complete.

Now, we establish the existence result of BVP(5) at non-resonance case, i.e., $R(C) = k$. The following abstract existence theorem will be used in the proof of the main result of this section. Its proof can be see in [10].

Lemma 2.2 [10]. Let X and Y be Banach spaces. Suppose $L : D(L) \subset X \to Y$ is a Fredholm operator of index zero with $\text{Ker}L = \{0\}$, $N : X \to Y$ is $L-$compact on each open bounded subset of X. If $0 \in \Omega \subset X$ is a open bounded subset and $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in (0, 1)$, then there exists at least one $x \in \Omega$ such that $Lx = Nx$.

Let $X = R^{2k+T}$ be endowed with the norm

$$
||x|| = \max_{n \in [0, 2k+T-1]} |x(n)|
$$

for $x = (x_0, x_1, \cdots, x_{2k+T-1}) \in R^{2k+T}$ and $Y = R^T \times R^k \times R^k$ be endowed with the norm

$$
||(x, u, v)|| = \max\left\{ \max_{n \in [0, T-1]} |x(n)|, \max_{i \in [0, k-1]} |u(i)|, \max_{i \in [k+T, 2k+T-1]} |v(i)| \right\}
$$

for $(x, u, v) \in Y$, where $x = (x_0, \cdots, x_{T-1}$, $u = (u_0, \cdots, u_{k-1})$ and $v = (v_0, \cdots, v_{k-1})$. It is easy to see that X and Y are Banach spaces.

Let $L : X \to Y$ be defined by

$$L(\ x\ ) = \begin{pmatrix} x(n+2k) - \sum_{i=0}^{2k-1} a_i x(n+i) \ \ n \in [0, T-1] \\ x(i) \ \ i \in [0, k-1] \\ x(i) \ \ i \in [T+k, T+2k-1] \end{pmatrix}, \quad x \in X,$$

and $N : X \to Y$ by

$$N(\ x\ ) = \begin{pmatrix} f(n, x(n), x(n+1), \cdots, x(n+2k-1)) \quad n \in [0, T-1] \\ A_i \qquad\qquad i \in [0, k-1] \\ B_i \qquad i \in [T+k, 2k+T-1] \end{pmatrix}, \quad x \in X.$$

Lemma 2.3. Suppose $R(C) = k$. Then it holds that

(i) $x \in X$ is a solution of $L(x) = N(x)$ implies that x is a solution of BVP(5).

(ii) $\text{Ker}L = \{0\}$.

(iii) L is a Fredholm operator of index zero, N is L-compact on each open bounded subset of X.

Proof. It is easy to see that (i) holds by the definitions of L and N. For (ii), suppose $x \in \text{Ker}L$, then we get (6). It follows that

$$\begin{aligned} x(k+T-1) &= C_{T-k-1} \begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix}, \\ x(k+T) &= C_{T-k} \begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix}, \\ &\cdots\cdots \\ x(2k+T-1) &= C_{T-1} \begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix}. \end{aligned}$$

Then (7) holds. Since $R(C) = k$, we get that

$$\begin{pmatrix} x(k) \\ x(k+1) \\ \cdots \\ x(2k-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}.$$

Hence $x(n) = 0$ for all $n \in [0, 2k+T-1]$.
The proof of (iii) is standard and is omitted, one may see [9,11]. The proof is complete.

Theorem 2.1. Suppose that $R(C) = k$, $a_k > 0$ and

(A) there exist numbers $\beta > 0$, $\theta > 1$, nonnegative sequences $p_i(n), r(n) (i \in [0, 2k-1])$, functions $g(n, x_0, \cdots, x_{2k-1})$, $h(n, x_0, \cdots, x_{2k-1})$ such that

$$f(n, x_0, \cdots, x_{2k-1}) = g(n, x_0, \cdots, x_{2k-1}) + h(n, x_0, \cdots, x_{2k-1}) \tag{8}$$

and

$$g(n, x_0, x_1, \cdots, x_{2k-1})x_{n+k} \geq \beta |x_k|^{\theta+1}, \tag{9}$$

and

$$|h(n, x_0, \cdots, x_{2k-1})| \leq \sum_{i=0}^{2k-1} p_i(n)|x_i|^{\theta} + r(n), \tag{10}$$

for all $n \in [0, T-1]$, $(x_0, x_1, \cdots, x_{2k-1}) \in R^{2k}$;

(B) it holds that

$$\sum_{j=0}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1} \frac{2ka_j^2}{a_k} + \frac{2k}{a_k} \leq 0. \tag{11}$$

Then BVP(5) has at least one solution if

$$\sum_{i=0}^{2k-1} ||p_i|| < \beta, \text{ where } ||p_i|| = \max_{n \in [0, T-1]} |p(n)|. \tag{12}$$

Proof. To apply Lemma 2.2, we consider $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in (0, 1)\}$. For $x \in \Omega_1$, we have

$$\begin{cases} x(n+2k) - \sum_{i=0}^{2k-1} a_i x(n+i) = \lambda f(n, x(n), x(n+1), \cdots, x(n+2k-1)), \\ \qquad\qquad n \in [0, T-1], \\ x(i) = \lambda A_i, \quad i \in [0, k-1], \\ x(i) = \lambda B_i, \quad i \in [T+k, T+2k-1]. \end{cases} \tag{13}$$

So

$$\left(x(n+2k) - \sum_{i=0}^{2k-1} a_i x(n+i)\right) x(n+k) = \lambda x(n+k) f(n, x(n), x(n+1), \cdots, x(n+2k-1)) \tag{14}$$

for all $n \in [0, T-1]$. It is easy to see that

$$\begin{aligned} &2\sum_{n=0}^{T-1} \left(x(n+2k) - \sum_{i=0}^{2k-1} a_i x(n+i)\right) x(n+k) \\ &= \sum_{n=0}^{T-1} \Big(2x(n+2k)x(n+k) - 2a_k[x(n+k)]^2 \\ &\quad -2\sum_{i \in [0,2k-1], i \neq k} a_i x(n+i)x(n+k)\Big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{T-1} \Bigg[-\left(\sqrt{\frac{2k}{a_k}} x(n+2k) - \sqrt{\frac{a_k}{2k}} x(n+k) \right)^2 \\
&\quad - \sum_{i=0}^{k-1} \left(\frac{a_i}{\sqrt{\frac{a_k}{2k}}} x(n+i) + \sqrt{\frac{a_k}{2k}} x(n+k) \right)^2 \\
&\quad - \sum_{i=k+1}^{2k-1} \left(\frac{a_i}{\sqrt{\frac{a_k}{2k}}} x(n+i) + \sqrt{\frac{a_k}{2k}} x(n+k) \right)^2 \\
&\quad -2a_k[x(n+k)]^2 + \frac{2k}{a_k}[x(n+2k)]^2 + \frac{a_k}{2k}[x(n+k)]^2 \\
&\quad + \sum_{i=0}^{k-1} \left(\frac{2ka_i^2}{a_k}[x(n+i)]^2 + \frac{a_k}{2k}[x(n+k)]^2 \right) \\
&\quad + \sum_{i=k+1}^{2k-1} \left(\frac{2ka_i^2}{a_k}[x(n+i)]^2 + \frac{a_k}{2k}[x(n+k)]^2 \right) \Bigg] \\
&\le \sum_{n=0}^{T-1} \Bigg[-a_k[x(n+k)]^2 + \frac{2k}{a_k}[x(n+2k)]^2 \\
&\quad + \sum_{i=0}^{k-1} \frac{2ka_i^2}{a_k}[x(n+i)]^2 + \sum_{i=k+1}^{2k-1} \frac{2ka_i^2}{a_k}[x(n+i)]^2 \Bigg] \\
&= \sum_{n=0}^{T-1} \Bigg[\sum_{i=0}^{k-1} \frac{2ka_i^2}{a_k}[x(n+i)]^2 - a_k[x(n+k)]^2 \\
&\quad + \sum_{i=k+1}^{2k-1} \frac{2ka_i^2}{a_k}[x(n+i)]^2 + \frac{2k}{a_k}[x(n+2k)]^2 \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2ka_0^2}{a_k} x(0)^2 + \left(\frac{2ka_0^2}{a_k} + \frac{2ka_1^2}{a_k} \right) x(1)^2 + \cdots + \left(\sum_{i=0}^{k-1} \frac{2ka_i^2}{a_k} \right) x(k-1)^2 \\
&\quad + \frac{2k}{a_k} x(2k+T-1)^2 + \left(\frac{2k}{a_k} + \frac{2ka_{2k-1}^2}{a_k} \right) x(2k+T-2)^2 + \cdots \\
&\quad + \left(\frac{2k}{a_k} + \sum_{i=T+k+1}^{2k-1} \frac{2ka_i^2}{a_k} \right) x(T+k)^2 \\
&\quad + \sum_{i=k}^{2k-1} \left(\sum_{j=0}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{i} \frac{2ka_j^2}{a_k} \right) x(i)^2 \\
&\quad + \sum_{i=2k}^{T-1} \left(\sum_{j=0}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1} \frac{2ka_j^2}{a_k} + \frac{2k}{a_k} \right) x(i)^2 \\
&\quad + \sum_{i=T}^{T+k-1} \left(\sum_{j=i-T+1}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1} \frac{2ka_j^2}{a_k} + \frac{2k}{a_k} \right) x(i)^2
\end{aligned}
$$

Since (11) and $a_k > 0$ imply that

$$
\begin{aligned}
&\sum_{j=0}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{i} \frac{2ka_j^2}{a_k} \le 0, \; i \in [k, 2k-1], \\
&\sum_{j=0}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1} \frac{2ka_j^2}{a_k} + \frac{2k}{a_k} \le 0,
\end{aligned}
$$

$$\sum_{j=i-T+1}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1} \frac{2ka_j^2}{a_k} + \frac{2k}{a_k} \leq 0,\ i \in [T, T+k-1].$$

Then

$$\begin{aligned}
& 2\sum_{n=0}^{T-1} \left(x(n+2k) - \sum_{i=0}^{2k-1} a_i x(n+i) \right) x(n+k) \\
\leq\ & \lambda^2 \left(\frac{2ka_0^2}{a_k} A_0^2 + \left(\frac{2ka_0^2}{a_k} + \frac{2ka_1^2}{a_k} \right) A_1^2 + \cdots + \left(\sum_{i=0}^{k-1} \frac{2ka_i^2}{a_k} \right) A_{k-1}^2 \right) \\
& + \lambda^2 \left(\frac{2k}{a_k} B_{2k+T-1}^2 + \left(\frac{2k}{a_k} + \frac{2ka_{2k-1}^2}{a_k} \right) B_{2k+T-2}^2 + \cdots \right. \\
& \qquad \left. + \left(\frac{2k}{a_k} + \sum_{i=T+k+1}^{2k-1} \frac{2ka_i^2}{a_k} \right) B_{T+k}^2 \right) \\
& + \sum_{i=k}^{2k-1} \left(\sum_{j=0}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{i} \frac{2ka_j^2}{a_k} \right) x(i)^2 \\
& \quad + \sum_{i=2k}^{T-1} \left(\sum_{j=0}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1} \frac{2ka_j^2}{a_k} + \frac{2k}{a_k} \right) x(i)^2 \\
& \quad + \sum_{i=T}^{T+k-1} \left(\sum_{j=i-T+1}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1} \frac{2ka_j^2}{a_k} + \frac{2k}{a_k} \right) x(i)^2 \\
\leq\ & \lambda^2 \left(\frac{2ka_0^2}{a_k} A_0^2 + \left(\frac{2ka_0^2}{a_k} + \frac{2ka_1^2}{a_k} \right) A_1^2 + \cdots + \left(\sum_{i=0}^{k-1} \frac{2ka_i^2}{a_k} \right) A_{k-1}^2 \right) \\
& \quad + \lambda^2 \left(\frac{2k}{a_k} B_{2k+T-1}^2 + \left(\frac{2k}{a_k} + \frac{2ka_{2k-1}^2}{a_k} \right) B_{2k+T-2}^2 + \cdots \right. \\
& \qquad \left. + \left(\frac{2k}{a_k} + \sum_{i=T+k+1}^{2k-1} \frac{2ka_i^2}{a_k} \right) B_{T+k}^2 \right) \\
=:\ & \lambda^2 M.
\end{aligned}$$

So, we get from (14) that

$$\lambda \sum_{n=0}^{T-1} x(n+k) f(n, x(n), x(n+1), \cdots, x(n+2k-1)) \leq \lambda^2 M.$$

It follows that

$$\sum_{n=0}^{T-1} x(n+k) f(n, x(n), x(n+1), \cdots, x(n+2k-1)) \leq \lambda M. \tag{15}$$

Then (8), (9) and (10) imply that

$$\begin{aligned}
& \beta \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \\
\leq \quad & \sum_{n=0}^{T-1} g(n, x(n), x(n+1), \cdots, x(n+2k-1)) x(n+k) \\
\leq \quad & \sum_{n=0}^{T-1} f(n, x(n), x(n+1), \cdots, x(n+2k-1)) x(n+k) \\
& - \sum_{n=0}^{T-1} h(n, x(n), x(n+1), \cdots, x(n+2k-1)) x(n+k) \\
\leq \quad & \lambda M - \sum_{n=0}^{T-1} h(n, x(n), x(n+1), \cdots, x(n+2k-1)) x(n+k) \\
\leq \quad & M + \sum_{n=0}^{T-1} |h(n, x(n), x(n+1), \cdots, x(n+2k-1))||x(n+k)|
\end{aligned}$$

$$\begin{aligned}
\leq \quad & M + \sum_{n=0}^{T-1} \left(\sum_{i=0}^{2k-1} p_i(n) |x(n+i)|^{\theta} + r(n) \right) |x(n+k)| \\
\leq \quad & \sum_{i=0}^{2k-1} ||p_i|| \sum_{n=0}^{T-1} |x(n+i)|^{\theta} |x(n+k)| \\
& + ||r|| \sum_{n=0}^{T-1} |x(n+k)| + M.
\end{aligned}$$

For $x_i \geq 0, y_i \geq 0 (i = 1, 2, \cdots, s)$, Holder$'$s inequality implies

$$\sum_{i=1}^{s} x_i y_i \leq \left(\sum_{i=1}^{s} x_i^p \right)^{1/p} \left(\sum_{i=1}^{s} y_i^q \right)^{1/q}, \quad 1/p + 1/q = 1, \quad q > 0, \ p > 0.$$

It follows that

$$\begin{aligned}
& \beta \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \\
& \leq ||p_k|| \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \\
& \quad + \sum_{i=0}^{k-1} ||p_i|| \left[\sum_{n=0}^{T-1} |x(n+i)|^{\theta+1} \right]^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \\
& \quad + \sum_{i=k+1}^{2k-1} ||p_i|| \left[\sum_{n=0}^{T-1} |x(n+i)|^{\theta+1} \right]^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \right]^{\frac{1}{\theta+1}} + M
\end{aligned}$$

$$
\begin{aligned}
= \quad & ||p_k|| \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \\
& + \left[\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \sum_{i=0}^{k-1} ||p_i|| \left[\sum_{u=i}^{T-1+i} |x(u)|^{\theta+1} \right]^{\frac{\theta}{\theta+1}} \\
& + \left[\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \sum_{i=k+1}^{2k-1} ||p_i|| \left[\sum_{u=i}^{T-1+i} |x(u)|^{\theta+1} \right]^{\frac{\theta}{\theta+1}} + M \\
\leq \quad & ||p_k|| \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \\
& + \left[\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \sum_{i=0}^{k-1} ||p_i|| \left[\sum_{u=i}^{T-1+k} |x(u)|^{\theta+1} \right]^{\frac{\theta}{\theta+1}} \\
& + \left[\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \sum_{i=k+1}^{2k-1} ||p_i|| \left[\sum_{u=k}^{T-1+i} |x(u)|^{\theta+1} \right]^{\frac{\theta}{\theta+1}} + M \\
= \quad & ||p_k|| \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \\
& + \left[\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \sum_{i=0}^{k-1} ||p_i|| \left[\sum_{u=k}^{T-1+k} |x(u)|^{\theta+1} + \sum_{u=i}^{k-1} |x(u)|^{\theta+1} \right]^{\frac{\theta}{\theta+1}} \\
& + \left[\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \sum_{i=k+1}^{2k-1} ||p_i|| \left[\sum_{u=k}^{T-1+k} |x(u)|^{\theta+1} + \sum_{u=T+k}^{T-1+i} |x(u)|^{\theta+1} \right]^{\frac{\theta}{\theta+1}} \\
& + M \\
= \quad & ||p_k|| \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \\
& + \left[\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \sum_{i=0}^{k-1} ||p_i|| \left[\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} + \sum_{u=i}^{k-1} |\lambda A_i|^{\theta+1} \right]^{\frac{\theta}{\theta+1}} \\
& + \left[\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \sum_{i=k+1}^{2k-1} ||p_i|| \left[\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} + \sum_{u=T+k}^{T-1+i} |\lambda B_i|^{\theta+1} \right]^{\frac{\theta}{\theta+1}} \\
& + M
\end{aligned}
$$

$$\leq \quad ||p_k||\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}+||r||T^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right]^{\frac{1}{\theta+1}}$$
$$+\left[\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}\right]^{\frac{1}{\theta+1}}\sum_{i=0}^{k-1}||p_i||\left[\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}+\sum_{u=0}^{k-1}|A_i|^{\theta+1}\right]^{\frac{\theta}{\theta+1}}$$
$$+\left[\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}\right]^{\frac{1}{\theta+1}}\sum_{i=k+1}^{2k-1}||p_i||\left[\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}+\sum_{u=T+k}^{T-1+2k}|B_i|^{\theta+1}\right]^{\frac{\theta}{\theta+1}}$$
$$+M$$

We claim that if $m>0$, then there exists a constant $\sigma\in(0,1)$, independent of λ, such that $(1+x)^m\leq 1+(m+1)x$ for all $x\in(0,\sigma)$. In fact, let $q(x)=(1+x)^m-(1+(m+1)x)$, we see $q(0)=0$, and $q'(0+)=-1<0$. Hence there exists a constant $\sigma>0$ such that $q'(x)<0$ for all $x\in[0,\sigma)$. Then $q(x)\leq q(0)=0$ for all $x\in[0,\sigma)$. It follows that the claim is valid. We consider two cases.

Case1. $\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}\leq\frac{\sum_{u=0}^{k-1}|A_i|^{\theta+1}+\sum_{u=T+k}^{T-1+2k}|B_i|^{\theta+1}}{\sigma}$.
At this case, we have

$$\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}\leq\frac{\sum_{u=0}^{k-1}|A_i|^{\theta+1}+\sum_{u=T+k}^{T-1+2k}|B_i|^{\theta+1}}{\sigma}=:M_1.$$

Case2. $\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}>\frac{\sum_{u=0}^{k-1}|A_i|^{\theta+1}+\sum_{u=T+k}^{T-1+2k}|B_i|^{\theta+1}}{\sigma}$.
At this case, one sees that

$$\sigma>\frac{\sum_{u=0}^{k-1}|A_i|^{\theta+1}+\sum_{u=T+k}^{T-1+2k}|B_i|^{\theta+1}}{\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}}.$$

Then

$$\sigma>\frac{\sum_{u=0}^{k-1}|A_i|^{\theta+1}}{\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}},\quad \sigma>\frac{\sum_{u=T+k}^{T-1+2k}|B_i|^{\theta+1}}{\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}}.$$

Hence

$$\beta\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}$$
$$\leq \quad ||p_k||\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}+||r||T^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T-1}|x(n+1)|^{\theta+1}\right]^{\frac{1}{\theta+1}}$$
$$+\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}\sum_{i=0}^{k-1}||p_i||\left[1+\frac{\sum_{u=0}^{k-1}|A_i|^{\theta+1}}{\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}}\right]^{\frac{\theta}{\theta+1}}$$
$$+\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}\sum_{i=k+1}^{2k-1}||p_i||\left[1+\frac{\sum_{u=T+k}^{T-1+2k}|B_i|^{\theta+1}}{\sum_{n=0}^{T-1}|x(n+k)|^{\theta+1}}\right]^{\frac{\theta}{\theta+1}}+M.$$

Using $(1+x)^m \leq 1+(m+1)x$ for all $x \in (0,\sigma)$, one gets that

$$\begin{aligned}
&\beta \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \\
\leq\ & ||p_k|| \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \\
& + \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \sum_{i=0}^{k-1} ||p_i|| \left[1 + \left(1+\frac{\theta}{1+\theta}\right) \frac{\sum_{u=0}^{k-1} |A_i|^{\theta+1}}{\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1}} \right] \\
& + \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \sum_{i=k+1}^{2k-1} ||p_i|| \left[1 + \left(1+\frac{\theta}{1+\theta}\right) \frac{\sum_{u=T+k}^{T-1+2k} |B_i|^{\theta+1}}{\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1}} \right] \\
& + M
\end{aligned}$$

$$\begin{aligned}
=\ & \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \sum_{i=0}^{2k-1} ||p_i|| + ||r|| T^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \\
& + \sum_{i=0}^{k-1} ||p_i|| \left(1+\frac{\theta}{1+\theta}\right) \sum_{u=0}^{k-1} |A_i|^{\theta+1} + \sum_{i=k+1}^{2k-1} ||p_i|| \left(1+\frac{\theta}{1+\theta}\right) \sum_{u=T+k}^{T-1+2k} |B_i|^{\theta+1} \\
& + M.
\end{aligned}$$

Then

$$\begin{aligned}
& \left[\beta - \sum_{i=0}^{2k-1} ||p_i|| \right] \sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \\
\leq\ & ||r|| T^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+1)|^{\theta+1} \right]^{\frac{1}{\theta+1}} \\
& + \sum_{i=0}^{k-1} ||p_i|| \left(1+\frac{\theta}{1+\theta}\right) \sum_{u=0}^{k-1} |A_i|^{\theta+1} + \sum_{i=k+1}^{2k-1} ||p_i|| \left(1+\frac{\theta}{1+\theta}\right) \sum_{u=T+k}^{T-1+2k} |B_i|^{\theta+1} \\
& + M.
\end{aligned}$$

It follows from (12) that there exists a constant $M_2 > 0$ such that

$$\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \leq M_2.$$

It follows from **Case 1** and **Case 2** that

$$\sum_{n=0}^{T-1} |x(n+k)|^{\theta+1} \leq \max\{M_1, M_2\}.$$

Hence $|x(n+k)| \leq (\max\{M_1, M_2\})^{\frac{1}{\theta+1}}$ for all $n \in [0, T-1]$. Hence

$$||x|| = \max_{n\in[0,T+2k-1]} |x(n)| \leq \max\left\{ \begin{array}{c} (\max\{M_1, M_2\})^{\frac{1}{\theta+1}},\ \max_{i\in[0,k-1]} |A_i|, \\ \max_{i\in[T+k,T+2k-1]} |B_i| \end{array} \right\}.$$

So Ω_1 is bounded.
Since f is continuous, **Lemma 2.3** implies that Ker$L = \{0\}$, we know that L is a Fredholm operator of index zero and N is $L-$compact on each open bounded subset of X. Let

$$\Omega_0 = \{x \in X : ||x|| < \max\left\{ \begin{array}{c} (\max\{M_1, M_2\})^{\frac{1}{\theta+1}},\ \max_{i\in[0,k-1]} |A_i|, \\ \max_{i\in[T+k,T+2k-1]} |B_i| \end{array} \right\} + 1\}.$$

Then $Lx \neq \lambda Nx$ for all $\lambda \in (0,1)$ and all $x \in D(L) \bigcap \partial\Omega_0$. It follows from **Lemma 2.2** that BVP (5) has at least one solution. The proof is complete.

Theorem 2.2. Suppose that $R(C) = k$, $a_k < 0$ and

(C) there exist numbers $\beta > 0$, $\theta > 1$, nonnegative sequences $p_i(n), r(n) (i = 0, \cdots, 2k-1)$, functions $g(n, x_0, \cdots, x_{2k-1})$, $h(n, x_0, \cdots, x_{2k-1})$ such that

$$f(n, x_0, \cdots, x_{2k-1}) = g(n, x_0, \cdots, x_{2k-1}) + h(n, x_0, \cdots, x_{2k-1}) \tag{16}$$

and

$$g(n, x_0, x_1, \cdots, x_{2k-1})x_{n+k} \leq -\beta|x_k|^{\theta+1}, \tag{17}$$

and

$$|h(n, x_0, \cdots, x_{2k-1})| \leq \sum_{i=0}^{2k-1} p_i(n)|x_i|^{\theta} + r(n), \tag{18}$$

for all $n \in [0, T-1]$, $(x_0, x_1, \cdots, x_{2k-1}) \in R^{2k}$;

(D) it holds that

$$\sum_{j=0}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{i} \frac{2ka_j^2}{a_k} \geq 0,\ i \in [k, 2k-1],$$
$$\sum_{j=0}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1} \frac{2ka_j^2}{a_k} + \frac{2k}{a_k} \geq 0,$$
$$\sum_{j=i-T+1}^{k-1} \frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1} \frac{2ka_j^2}{a_k} + \frac{2k}{a_k} \geq 0,\ i \in [T, T+k-1].$$

Then BVP(5) has at least one solution if

$$\sum_{i=0}^{2k-1} ||p_i|| < \beta, \text{ where } ||p_i|| = \max_{n\in[0,T-1]} |p(n)|. \tag{19}$$

Proof. To apply Lemma 2.2, we consider $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in (0,1)\}$. For $x \in \Omega_1$, we get (13) and (14). It is easy to see that

$$
\begin{aligned}
& 2\sum_{n=0}^{T-1}\left(x(n+2k) - \sum_{i=0}^{2k-1} a_i x(n+i)\right)x(n+k) \\
= & \sum_{n=0}^{T-1}\Big(2x(n+2k)x(n+k) - 2a_k[x(n+k)]^2 \\
& \qquad -2\sum_{i\in[0,2k-1],i\neq k} a_i x(n+i)x(n+k)\Big) \\
= & \sum_{n=0}^{T-1}\left[\left(\sqrt{\frac{2k}{-a_k}}x(n+2k) + \sqrt{\frac{-a_k}{2k}}x(n+k)\right)^2\right. \\
& + \sum_{i=0}^{k-1}\left(\frac{a_i}{\sqrt{\frac{-a_k}{2k}}}x(n+i) - \sqrt{\frac{-a_k}{2k}}x(n+k)\right)^2 \\
& + \sum_{i=k+1}^{2k-1}\left(\frac{a_i}{\sqrt{\frac{-a_k}{2k}}}x(n+i) - \sqrt{\frac{-a_k}{2k}}x(n+k)\right)^2 \\
& - 2a_k[x(n+k)]^2 + \frac{2k}{a_k}[x(n+2k)]^2 + \frac{a_k}{2k}[x(n+k)]^2 \\
& + \sum_{i=0}^{k-1}\left(\frac{2ka_i^2}{a_k}[x(n+i)]^2 + \frac{a_k}{2k}[x(n+k)]^2\right) \\
& \left. + \sum_{i=k+1}^{2k-1}\left(\frac{2ka_i^2}{a_k}[x(n+i)]^2 + \frac{a_k}{2k}[x(n+k)]^2\right)\right] \\
\geq & \sum_{n=0}^{T-1}\left[-a_k[x(n+k)]^2 + \frac{2k}{a_k}[x(n+2k)]^2\right. \\
& \left. + \sum_{i=0}^{k-1}\frac{2ka_i^2}{a_k}[x(n+i)]^2 + \sum_{i=k+1}^{2k-1}\frac{2ka_i^2}{a_k}[x(n+i)]^2\right] \\
= & \sum_{n=0}^{T-1}\left[\sum_{i=0}^{k-1}\frac{2ka_i^2}{a_k}[x(n+i)]^2 - a_k[x(n+k)]^2 + \sum_{i=k+1}^{2k-1}\frac{2ka_i^2}{a_k}[x(n+i)]^2\right. \\
& \left. + \frac{2k}{a_k}[x(n+2k)]^2\right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{2ka_0^2}{a_k}x_0^2 + \left(\frac{2ka_0^2}{a_k} + \frac{2ka_1^2}{a_k}\right)x_1^2 + \cdots + \left(\sum_{i=0}^{k-1}\frac{2ka_i^2}{a_k}\right)x_{k-1}^2 \\
& + \frac{2k}{a_k}x_{2k+T-1}^2 + \left(\frac{2k}{a_k} + \frac{2ka_{2k-1}^2}{a_k}\right)x_{2k+T-2}^2 + \cdots \\
& + \left(\frac{2k}{a_k} + \sum_{i=T+k+1}^{2k-1}\frac{2ka_i^2}{a_k}\right)x_{T+k}^2 + \sum_{i=k}^{2k-1}\left(\sum_{j=0}^{k-1}\frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{i}\frac{2ka_j^2}{a_k}\right)x_i^2 \\
& + \sum_{i=2k}^{T-1}\left(\sum_{j=0}^{k-1}\frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1}\frac{2ka_j^2}{a_k} + \frac{2k}{a_k}\right)x_i^2 \\
& + \sum_{i=T}^{T+k-1}\left(\sum_{j=i-T+1}^{k-1}\frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1}\frac{2ka_j^2}{a_k} + \frac{2k}{a_k}\right)x_i^2
\end{aligned}
$$

Since (D) and $a_k < 0$ imply that

$$
\begin{aligned}
& 2\sum_{n=0}^{T-1}\left(x(n+2k) - \sum_{i=0}^{2k-1}a_i x(n+i)\right)x(n+k) \\
\geq \quad & \lambda^2\left(\frac{2ka_0^2}{a_k}A_0^2 + \left(\frac{2ka_0^2}{a_k} + \frac{2ka_1^2}{a_k}\right)A_1^2 + \cdots + \left(\sum_{i=0}^{k-1}\frac{2ka_i^2}{a_k}\right)A_{k-1}^2\right) \\
& + \lambda^2\left(\frac{2k}{a_k}B_{2k+T-1}^2 + \left(\frac{2k}{a_k} + \frac{2ka_{2k-1}^2}{a_k}\right)B_{2k+T-2}^2 + \cdots \right. \\
& \left. + \left(\frac{2k}{a_k} + \sum_{i=T+k+1}^{2k-1}\frac{2ka_i^2}{a_k}\right)B_{T+k}^2\right) \\
& + \sum_{i=k}^{2k-1}\left(\sum_{j=0}^{k-1}\frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{i}\frac{2ka_j^2}{a_k}\right)x_i^2 \\
& + \sum_{i=2k}^{T-1}\left(\sum_{j=0}^{k-1}\frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1}\frac{2ka_j^2}{a_k} + \frac{2k}{a_k}\right)x_i^2 \\
& + \sum_{i=T}^{T+k-1}\left(\sum_{j=i-T+1}^{k-1}\frac{2ka_j^2}{a_k} - a_k + \sum_{j=k+1}^{2k-1}\frac{2ka_j^2}{a_k} + \frac{2k}{a_k}\right)x_i^2 \\
\geq \; & \lambda^2\left(\tfrac{2ka_0^2}{a_k}A_0^2 + \left(\tfrac{2ka_0^2}{a_k} + \tfrac{2ka_1^2}{a_k}\right)A_1^2 + \cdots + \left(\textstyle\sum_{i=0}^{k-1}\tfrac{2ka_i^2}{a_k}\right)A_{k-1}^2\right) \\
& + \lambda^2\left(\tfrac{2k}{a_k}B_{2k+T-1}^2 + \left(\tfrac{2k}{a_k} + \tfrac{2ka_{2k-1}^2}{a_k}\right)B_{2k+T-2}^2 + \cdots \right. \\
& \left. + \left(\tfrac{2k}{a_k} + \textstyle\sum_{i=T+k+1}^{2k-1}\tfrac{2ka_i^2}{a_k}\right)B_{T+k}^2\right) \\
=: \; & \lambda^2 M.
\end{aligned}
$$

So, we get from (14) that

$$\lambda \sum_{n=0}^{T-1} x(n+k) f(n, x(n), x(n+1), \cdots, x(n+2k-1)) \geq \lambda^2 M.$$

Hence

$$\sum_{n=0}^{T-1} x(n+k) f(n, x(n), x(n+1), \cdots, x(n+2k-1)) \geq M.$$

The remainder of the proof is similar to that of the proof of Theorem L1 and is omitted. The proof is completed.

3. Examples

In this section, we present two examples to illustrate the main result in section 2.

Example 3.1. Consider the following BVP
(20)

$$\begin{cases} x(n+2) - 3x(n+1) - x(n) \\ \quad = \beta[x(n+1)]^{2m+1} + p_0(n)[x(n)]^{2m+1} + p_1(n)[x(n+1)]^{2m+1} + r(n), \\ \qquad n \in [0, T-1], \\ x(0) = A, \\ x(T+1) = B, \end{cases}$$

where $T \geq 2$ is a positive integer, $m \geq 0$, $r > 1$ integers, $\beta > 0$, $p_0(n), p_1(n), r(n)$ are sequences. Corresponding to BVP(5), we set $k = 1$, $A_0 = A, B_0 = B$, $a_0 = 1, a_1 = 3 > 0$, and

$$f(n, x_0, x_1) = \beta x_1^{2m+1} + p_0(n) x_0^{2m+1} + p_1(n) x_1^{2m+1} + r(n),$$

$$g(n, x_0, x_1) = \beta x_0^{2m+1},$$

and

$$h(n, x_0, x_1 n) = p_0(n) x_0^{2m+1} + p_1(n) x_i^{2m+1} + r(n).$$

One sees that **(A)** and **(B)** in **Theorem 2.1** hold. It is easy to see that

$$\begin{aligned} C_0 &= (a_1) = (3), \\ C_1 &= a_1 C_0 + (1) = (4), \\ C_2 &= a_1 C_1 + a_0 C_0 = (13), \\ C_3 &= a_1 C_2 + a_0 C_1, \\ &\cdots\cdots \\ C_{T-1} &= a_1 C_{T-2} + a_0 C_{T-3}. \end{aligned}$$

One see that $R(C) = 1$. Then all assumptions in **Theorem 2.1** hold. It follows from **Theorem 2.1** that BVP(20) has at least one solution if

$$||p_0|| + ||p_1|| < \beta.$$

Example 3.2. Consider the following BVP
(21)

$$\begin{cases} x(n+4) - x(n) - 2x(n+1) + 3x(n+2) - 4x(n+3) \\ \qquad\qquad = -\beta[x(n+2)]^3 + \sum_{i=0}^{3} p_i(n)[x(n+i)]^3 + r(k), n \in [0, T-1], \\ x(0) = A_0, x(1) = A_1, \\ x(T+2) = B_0, x(T+3) = B_1, \end{cases}$$

where$T \geq 4$ is a positive integer, $\beta > 0$, $p_0(n), p_i(n), r(n)$ are sequences. Corresponding to BVP(5), we set $a_0 = 1, a_1 = 2, a_2 = -3 < 0, a_3 = 4$, $k = 2$, and

$$f(n, x_0, x_1, x_2, x_3) = -\beta x_2^3 + p_0(n)x_0^3 + p_1(n)x_1^3 + p_2(n)x_2^3 + p_3(n)x_3^3 + r(n),$$

$$g(n, x_0, x_1, x_2, x_3) = -\beta x_2^3,$$

and

$$h(n, x_0, x_1, x_2, x_3) = p_0(n)x_0^3 + p_1(n)x_1^3 + p_2(n)x_2^3 + p_3(n)x_3^3 + r(n).$$

It is easy to see that **(C)** in **Theorem 2.2** hold. One can get C by computation and finds that $R(C) = 2$. Hence **(D)** in **Theorem 2.2** holds. It follows from **Theorem 2.2** that BVP(21) has at least one solution if

$$||p_0|| + ||p_1|| + ||p_2|| + ||p_3|| < \beta.$$

References

1. R.P. Agarwal, Focal Boundary Value Problems for Differential and Difference Equations, Kluwer, Dordrecht, 1998.
2. R.P. Agarwal, D.O'Regan, P.J.Y. Wong, Positive Solutions of Differential, Difference and Integral Equations, Kluwer Academic Publishers, Dordrecht, 1999.
3. R.P. Agarwal, On multipoint boundary value problems for discrete equations, J. Math. Anal. Appl. 96(2)(1983)520-534.
4. I. Rachnkovab, C.C. Tisdell, Existence of non-spuurious solutions to discrete boundary value problems, The Australian Journal of Mathematical Analysis and Applications, 2006(3)2:1-9.
5. H.B. Thompson, C.C. Tisdell, The nonexistence of spurious solutions to discrete, two-point boundary value problems, Appl. Math. Lett. 16(1)(2003)79-84.
6. I. Rachonkova, C. Tisdell, Existence of non-spurious solutions to discrete Dirichlet problems with lower and upper solutions, Nonl. Anal., 67(2007)1236-1245.
7. C.C. Tisdell, The uniqueness of solutions to discrete, vector, two-point boundary value problems, Appl. Math. Lett. 16(8)(2003)1321-1328.
8. A. Cabada, V. Otero-Espinar, Fixed sign solutions of second-order difference equations with Neumann boundary conditions, Comput. Math. Appl. .45(2003)1125-1136.
9. Y. Liu, Periodic solutions of nonlinear functional difference equations at nonresonance case, J. Math. Anal. Appl., 327(2007)801-815.

10. R. E. Gaines, J. L. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Lecture Notes in Math., 568, Springer, Berlin, 1977.
11. R. I. Avery, Chuan Jen Chyan, J. Henderson, Twin solutions of boundary value problems for ordinary differential equations and finite difference equations, Comput. Math. Appl. 42(2001)695-704.
12. P.W. Eloe, J. Henderson and E. Kaufmann, Multiple positive solutions for difference equations, J. Difference Equations and Applications 3(1998)219-229.
13. F. Merdivenci, Two positive solutions for a boundary value problem for difference equations, J. Difference Equations and Applications 1(1995)262-270.
14. A. Cabada, N. D. Dimitrov, Multiplicity results for nonlinear periodic fourth order difference equations with parameter dependence and singularities, Journal of Mathematical Analysis and Applications, In Press, doi:10.1016/j.jmaa.2010.05.052.