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Generalized Leibniz Rule for an Extended Fractional Derivative Operator with Applications to Special Functions

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Abstract. Recently an extended operator of fractional derivative related to a generalized beta function has been used in order to obtain some generating relations involving extended hypergeometric functions [19]. In this paper, an extended fractional derivative operator with respect to an arbitrary regular and univalent function based on the Cauchy integral formula is defined. This is done to compute the extended fractional derivative of the function $\log z$ and principally, to obtain a generalized Leibniz rule. Some examples involving special functions are given. A representation of the extended fractional derivative operator in terms of the classical fractional derivative operator is also determined by using a result of A.R. Miller [12].

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1. Introduction

Several extensions of special functions have been obtained recently by several authors [4, 5, 6, 7, 8, 13]. Especially, Chaudhry et al. [4] gave an extension of the Euler's beta function. Namely, they defined the following extended beta function

$$B_p(x,y) = B(x,y;p) := \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[\frac{-p}{t(1-t)}\right] dt \tag{1.1}$$

which is valid for $\mathrm{Re}(p) > 0$. The case $p = 0$ gives the common beta function requiring that $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$. They also defined the extended hypergeometric function in [5] as follows

(1.2)
$$F_p(a,b;c;z) = \sum_{n=0}^{\infty} \frac{B(b+n, c-b; p)(a)_n}{B(b, c-b)} \frac{z^n}{n!}; \;\; p \geq 0; \; |z| < 1; \; \mathrm{Re}(c) > \mathrm{Re}(b) > 0,$$

where $(\lambda)_\nu$ denotes Pochhammer's symbol defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}; \quad (\lambda)_0 = 1$$

and they obtained the corresponding Euler type integral representation

$$\text{(1.3)} \;\; F_p(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left[\frac{-p}{t(1-t)}\right] dt$$

with $p > 0$; $p = 0$ and $|\arg(1-z)| < \pi$; $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$.
Very recently, using the well-known Riemann-Liouville integral representation for fractional derivative

$$D_z^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z \frac{f(\zeta)\mathrm{d}\zeta}{(z-\zeta)^{\alpha+1}} \tag{1.4}$$

which is valid for $\mathrm{Re}(\alpha) < 0$, where the integration path is a line from 0 to z in the complex ζ-plane and where the case $m-1 < \mathrm{Re}(\alpha) < m$ $(m = 1, 2, 3, ...)$ yields

$$D_z^\alpha f(z) = \frac{d^m}{dz^m} D_z^{\alpha-m} f(z) = \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(-\alpha+m)} \int_0^z \frac{f(\zeta)\mathrm{d}\zeta}{(z-\zeta)^{\alpha-m+1}}\right\},$$

Özarslan and Özergin [19] defined the following extended Riemann-Liouville fractional derivative by adding a new parameter. Explicitly, they considered

$$D_z^{\alpha,p} f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z f(\zeta)(z-\zeta)^{-\alpha-1} \exp\left[\frac{-pz^2}{\zeta(z-\zeta)}\right] \mathrm{d}\zeta \tag{1.5}$$

with $\mathrm{Re}(\alpha) < 0$, $\mathrm{Re}(p) > 0$ and for $m-1 < \mathrm{Re}(\alpha) < m$ $(m = 1, 2, 3, ...)$,

$$D_z^{\alpha,p} f(z) = \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(-\alpha+m)} \int_0^z f(\zeta)(z-\zeta)^{-\alpha+m-1} \exp\left[\frac{-pz^2}{\zeta(z-\zeta)}\right] \mathrm{d}\zeta\right\}.$$

The path of integration is a line from 0 to z in the complex ζ-plane. It is easy to see that the case $p = 0$ gives the classical Riemann-Liouville fractional derivative operator. Using this definition, they calculated the extended fractional derivatives for some elementary functions. Here are some of them.

Case 1. (See, [19, Theorem 3.1, p. 1828].) Let $\mathrm{Re}(\lambda) > -1$, $\mathrm{Re}(\alpha) < 0$. Then

$$D_z^{\alpha,p} z^{\lambda} = \frac{B(\lambda+1, -\alpha; p)}{\Gamma(-\alpha)} z^{\lambda-\alpha}. \tag{1.6}$$

Case 2. (See, [19, Theorem 3.2, p.1829].) Let $\mathrm{Re}(\lambda) > 0$, $\mathrm{Re}(\alpha) < 0$, $\mathrm{Re}(\mu) > 0$ and $|z| < 1$. Then

$$D_z^{\lambda-\alpha,p} z^{\lambda-1}(1-z)^{-\mu} = \frac{\Gamma(\lambda)}{\Gamma(\alpha)} z^{\alpha-1} F_p(\mu, \lambda; \alpha; z). \tag{1.7}$$

Furthermore, they also defined the extended Appell's hypergeometric functions of two variables $F_1(a, b, c; d; x, y; p)$ and $F_2(a, b, c; d, e; x, y, p)$. Namely,

$$F_1(a, b, c; d; x, y; p) := \sum_{n,m=0}^{\infty} \frac{B(a+m+n, d-a; p)}{B(a, d-a)} (b)_n (c)_m \frac{x^n}{n!} \frac{y^m}{m!}, \quad \max\{|x|, |y|\} < 1, \tag{1.8}$$

and

$$F_2(a, b, c; d, e; x, y; p) := \sum_{n,m=0}^{\infty} \frac{(a)_{m+n} B(b+n, d-b; p) B(c+m, e-c; p)}{B(b, d-b) B(c, e-c)} \frac{x^n}{n!} \frac{y^m}{m!}, \quad |x|+|y| < 1. \tag{1.9}$$

Here again, the case $p = 0$ gives the familiar functions. They also obtained their integral representation and showed the connection between these functions and the extended Riemann-Liouville fractional derivative operator. As example, they got

Case 3. (See, [19, Theorem 3.4, p.1830].) For $\mathrm{Re}(\mu) > \mathrm{Re}(\lambda) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Ree}(\gamma) > 0$; $\left|\frac{x}{1-z}\right| < 1$ and $|x| + |y| < 1$, we have

$$D_z^{\lambda-\mu,p} z^{\lambda-1}(1-z)^{-\alpha} F_p\left(\alpha, \beta; \gamma; \frac{x}{1-z}\right) = \frac{1}{B(\beta, \gamma-\beta)\Gamma(\mu-\lambda)} z^{\mu-1} F_2(\alpha, \beta, \lambda; \gamma, \mu; x, z; p). \tag{1.10}$$

More generally, if we consider an analytic function $f(z)$ in the disk $|z| < \rho$ with the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have ([19, Theorem 3.5, p.1830])

$$D_z^{\mu,p} z^{\lambda-1} f(z) = \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n B(\lambda+n, -\mu) z^n \tag{1.11}$$

provided that $\mathrm{Re}(\lambda) > 0$, $\mathrm{Re}(\mu) < 0$ and $|z| < \rho$.

In 1970, considering a fractional derivative representation based on the Cauchy integral formula, Osler [16] obtained the following generalized Leibniz rule for fractional derivatives

$$D_z^{\alpha} z^{p+q} u(z)v(z) = \sum_{n=-\infty}^{\infty} \begin{pmatrix} \alpha \\ \gamma+n \end{pmatrix} D_z^{\alpha-\gamma-n} z^p u(z) D_z^{\gamma+n} z^q v(z) \tag{1.12}$$

which yields for α non-negative integer, γ an arbitrary complex number, $\mathrm{Re}(p) > -1$, $\mathrm{Re}(q) > -1$ and $\mathrm{Re}(p+q) > -1$. Numerous interesting applications of this rule have been given. In particular, the Leibniz rule has been effective in the summation of infinite series [10, 11, 15, 16, 18, 22].
The aim of this paper is to present a generalized Leibniz rule for the extended fractional derivative operator and to give some applications in the summation of infinite series. Firstly, in section 2, we give a representation based on Cauchy integral formula for the extended fractional derivative operator, we define the extended fractional derivative with respect to an arbitrary regular and univalent function, we calculate the extended fractional derivative of the function log z and we determine a representation of the extended fractional derivative operator in terms of the classical fractional derivative operator. In section 3, we establish the generalized Leibniz rule for this operator. Finally, section 4 is devoted to applications of this new Leibniz rule.

2. Extended Fractional Derivative Operator Based on Cauchy Integral Formula

The derivative of arbitrary order $\alpha \in C$ related with the Cauchy integral formula [1, 2, 3, 14, 16, 17] is defined by

$$D_z^{\alpha} z^{\lambda} f(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_0^{(z+)} f(\zeta)\zeta^{\lambda}(\zeta - z)^{-\alpha-1} \mathrm{d}\zeta \tag{2.1}$$

where the contour is shown in figure 1 and consists of a single loop beginning at $\zeta = 0$ enclosing the point $\zeta = z$ once in the positive direction and returns to $\zeta = 0$ without cutting the branch line for $\zeta^{\lambda}(\zeta - z)^{-\alpha-1}$ which is valid for α not a negative integer and $\mathrm{Re}(\lambda) > -1$. This definition for the fractional derivative has been very effective in obtaining very interesting new results. We will adopt here the following definition for the extended fractional derivative operator

$$D_z^{\alpha,p} z^{\lambda} f(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_0^{(z+)} f(\zeta)\zeta^{\lambda}(\zeta - z)^{-\alpha-1} \exp\left[\frac{-pz^2}{\zeta(z-\zeta)}\right] \mathrm{d}\zeta \tag{2.2}$$

where the contour is shown in figure 1 with $\mathrm{Re}(p) > 0$, α not a negative integer and $\mathrm{Re}(\lambda) > -1$. It is obvious that the case $p = 0$ gives the classical fractional derivative operator (2.1). This new definition will be of great importance in order to obtain a Leibniz rule for this extended operator.

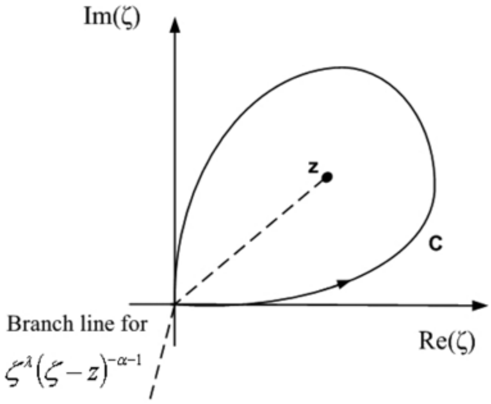

Figure 1. Single loop contour

Moreover, we can consider an extended fractional derivative operator with respect to an arbitrary regular and univalent function $g(z)$ as Osler did in [16]. Precisely, we have the following more general definition for the extended fractional derivative operator

$$D^{\alpha,p}_{g(z)} g\left(z\right)^{\lambda} f\left(z\right) = \tfrac{\Gamma(1+\alpha)}{2\pi i} \int_{g^{-1}(0)}^{(z+)} w\left(\zeta\right) \mathrm{d}\zeta \tag{2.3}$$

such that

$$w\left(\zeta\right) = f\left(\zeta\right) g\left(\zeta\right)^{\lambda} \left(g\left(\zeta\right) - g\left(z\right)\right)^{-\alpha-1} \exp\left[\tfrac{-pg(z)^2}{g(\zeta)g(z)-g(\zeta)}\right] g'\left(\zeta\right)$$

where the contour is now starting at $g^{-1}(0)$ encircles z in the positive sense and returns to $g^{-1}(0)$ without enclosing singularities of $f(z)$. This definition is valid for $Re(p) > 0$, α not a negative integer, $Re(\lambda) > -1$ and $z \neq g^{-1}(0)$. Letting $g(\zeta) = ug(z)$ in (2.3), we get the equivalent form

(2.4)

$$D^{\alpha,p}_{g(z)} g(z)^{\lambda} f(z) = \tfrac{\Gamma(1+\alpha)g(z)^{\lambda-\alpha}}{2\pi i} \int_{0}^{(1+)} f(g^{-1}(ug(z)))u^{\lambda}(u-1)^{-\alpha-1} \exp\left[\tfrac{-p}{u(1-u)}\right] \mathrm{d}u.$$

This definition suggests a generalization of all the special functions that possess a representation in terms of the fractional derivatives (see table 1) by simply replacing the classical fractional derivative operator by the extended one in the representation.

Table 1

Some special functions expressed in terms of fractional derivatives

Name	Fractional derivative representation
Gauss hypergeometric function	${}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)} z^{1-\gamma} D_z^{\beta-\gamma} z^{\beta-1}(1-z)^{-\alpha}$
Degenerate hypergeometric function	${}_1F_1(\alpha;\beta;z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} z^{1-\beta} D_z^{\alpha-\beta} z^{\alpha-1} \mathrm{e}^z$
Generalized hypergeometric function	${}_{p+1}F_{q+1}(\alpha, a_1, ..., a_p; \gamma, b_1, ..., b_q; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} z^{1-\gamma} D_z^{\alpha-\gamma} z^{\alpha-1} \; {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z)$
Bessel function	$J_\mu(z) = \frac{z^{-\mu}}{2^{\mu-1}\sqrt{\pi}} D_{z^2}^{-\mu+1/2} \sin z = \frac{z^{-\mu}}{2^{\mu}\sqrt{\pi}} D_{z^2}^{-\mu-1/2} \frac{\cos z}{z} = \frac{z^{-\mu}}{2^{\mu-\nu}} D_{z^2}^{-\mu+\nu} z^\nu J_\nu(z)$
Modified Bessel function	$I_\mu(z) = \frac{z^{-\mu}}{2^{\mu}\sqrt{\pi}} D_{z^2}^{-\mu-1/2} \frac{\cosh z}{z}$
Struve function	$H_\mu(z) = \frac{z^{-\mu}}{2^{\mu}\sqrt{\pi}} D_{z^2}^{-\mu-1/2} \frac{\sin z}{z}$
Modified Struve function	$L_\mu(z) = \frac{z^{-\mu}}{2^{\mu}\sqrt{\pi}} D_{z^2}^{-\mu-1/2} \frac{\sinh z}{z}$
Legendre function of the 1st kind	$P_\mu(z) = \frac{1}{\Gamma(1+\mu)2^\mu} D_{1-z}^{\mu} (1-z^2)^\mu$
Associated Legendre function (1st kind)	$P_\nu^\mu(z) = \frac{(1-z^2)^{\mu/2}}{\Gamma(1+\nu)2^\nu} D_{1-z}^{\mu+\nu} (1-z^2)^\nu$
Associated Legendre function (2nd kind)	$Q_\mu^\nu(z) = \frac{\mathrm{e}^{i\pi\mu}\sqrt{\pi}\Gamma(1+\mu+\nu)\Gamma(-\mu-\nu)}{\Gamma(-1-2\nu)\Gamma(3/2+\nu)2^{\nu+1}} (1-z^2)^{\mu/2} D_{1-z}^{-1+\mu-\nu} (1-z^2)^{-1-\nu}$
Jacobi function	$P_\mu^{(\alpha,\beta)}(z) = \frac{\Gamma(1+\alpha+\mu)}{2^\mu\Gamma(1+\mu)\Gamma(1+\alpha+\beta+\mu)} (1-z)^{-\alpha} D_{1-z}^{\beta+\mu} (1-z)^{\alpha+\beta+\mu}(1+z)^\mu$
	$P_\mu^{(\alpha,\beta)}(z) = \frac{\Gamma(1+\beta+\mu)\mathrm{e}^{i\pi\mu}}{2^\mu\Gamma(1+\mu)\Gamma(1+\alpha+\beta+\mu)} (1+z)^{-\beta} D_{1+z}^{\alpha+\mu} (1+z)^{\alpha+\beta+\mu}(1-z)^\mu$
	$P_\mu^{(\alpha,\beta)}(z) = \frac{(z-1)^{-\alpha}(z+1)^{-\beta}}{2^\mu\Gamma(1+\mu)} D_{1+z}^{\mu} (z-1)^{\alpha+\mu}(z+1)^{\beta+\mu}$
Laguerre function	$L_\mu^{(\alpha)}(z) = \frac{z^{-\alpha}}{\Gamma(1+\mu)} \mathrm{e}^z D_z^\mu z^{\alpha+\mu} \mathrm{e}^{-z} = \frac{\Gamma(1+\mu+\alpha)}{\Gamma(1+\mu)\Gamma(-\mu)} z^{-\alpha} D_z^{-\alpha-\mu-1} z^{-\mu-1} \mathrm{e}^z$
Incomplete gammma function	$\gamma(\alpha,z) = \Gamma(\alpha)\mathrm{e}^{-z} D_z^{-\alpha} \mathrm{e}^z$
Psi function	$\Psi(\xi) = -\gamma + \ln(z) - \Gamma(\xi) z^{1-\xi} D_z^{1-\xi} \ln(z)$
Whittaker function	$M_{\mu;\nu}(z) = \frac{\Gamma(1+2\nu)}{\Gamma(1/2+\nu-\mu)} \mathrm{e}^{-z/2} z^{-3/2-\nu} D_z^{-1/2-\nu-\mu} z^{\nu-\mu-1/2} \mathrm{e}^z$

Example 2.1. Extended Legendre function of the first kind

If we set $g(z) = 1 - z$, $f(z) = (1+z)^\mu$, $\lambda = \mu$ and $\alpha = \mu$ in (2.4) and if we divide by $\Gamma(1+\mu)$, we get the following extension of the Legendre function of the first kind in terms of the extended fractional derivative operator

$$P_{\mu,p}(z) = \frac{1}{\Gamma(1+\mu)\, 2^{\mu}} D^{\mu,p}_{1-z} \left(1 - z^{2}\right)^{\mu}. \tag{2.5}$$

Explicitly, we have

(2.6)

$$\begin{aligned}
P_{\mu,p}(z) &= \tfrac{1}{\Gamma(1+\mu)2^{\mu}} \tfrac{\Gamma(1+\mu)}{2\pi i} \int_0^{(1+)} [2 - t(1-z)]^{\mu}\, t^{\mu}(t-1)^{-\mu-1} \exp\left[\tfrac{-p}{t(1-t)}\right] \mathrm{d}t \\
&= \tfrac{1}{\Gamma(1+\mu)} \tfrac{\Gamma(1+\mu)}{2\pi i} \int_0^{(1+)} \left[1 - \tfrac{t(1-z)}{2}\right]^{\mu} t^{\mu}(t-1)^{-\mu-1} \exp\left[\tfrac{-p}{t(1-t)}\right] \mathrm{d}t \\
&= \tfrac{1}{\Gamma(1+\mu)} \tfrac{\Gamma(1+\mu)}{2\pi i} \int_0^{(1+)} \sum_{n=0}^{\infty} \tfrac{(-\mu)_n}{n!} \left(\tfrac{1-z}{2}\right)^{n} t^{\mu+n}(t-1)^{-\mu-1} \exp\left[\tfrac{-p}{t(1-t)}\right] \mathrm{d}t \\
&= \tfrac{1}{\Gamma(1+\mu)} \sum_{n=0}^{\infty} \tfrac{(-\mu)_n}{n!} \left(\tfrac{1-z}{2}\right)^{n} \tfrac{\Gamma(1+\mu)}{2\pi i} \int_0^{(1+)} t^{\mu+n}(t-1)^{-\mu-1} \exp\left[\tfrac{-p}{t(1-t)}\right] \mathrm{d}t \\
&= \tfrac{1}{\Gamma(1+\mu)} \sum_{n=0}^{\infty} \tfrac{(-\mu)_n}{n!} \left(\tfrac{1-z}{2}\right)^{n} \tfrac{B(\mu+n+1,-\mu;p)}{\Gamma(-\mu)}.
\end{aligned}$$

The last equality is obtained with the help of the following integral representation for the extended beta function

$$B(x,y;p) = \frac{1}{2i\sin\pi y} \int_0^{(1+)} t^{\,x-1}(t-1)^{\,y-1} \exp\left[\frac{-p}{t(1-t)}\right] \mathrm{d}t \tag{2.7}$$

which yields for $\mathrm{Re}(x) > 0$, y not an integer and $\mathrm{Re}(p) > 0$.
Making use of (1.2), we can rewrite the last equality in (2.6) in terms of the extended hypergeometric function. Thus, we have

$$P_{\mu,p}(z) = F_p\left(-\mu, \mu+1; 1; \frac{1-z}{2}\right) \tag{2.8}$$

valid for μ not a negative integer and $\mathrm{Re}(p) > 0$. It is easy to see that the case $p = 0$ gives the classical representation of the Legendre function of the first kind in terms of hypergeometric function (23, eq. 29, p. 34).
Another interesting formula is the extended fractional derivative of the function $\log z$. In 1998 A.R. Miller in [12, eq. (3.2), p. 29] showed that for $\mathrm{Re}(x) > -1$, $\mathrm{Re}(y) > -1$ and $\mathrm{Re}(p) > 0$, we have

$$B(x,y;p) = \mathrm{e}^{-2p} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B(x+m+1, y+n+1) L_m(p) L_n(p) \tag{2.9}$$

where $L_m(p)$ denotes the well-known classical Laguerre polynomials defined by the generating relation (see [20, eq. 4, p. 202])

$$(1-t)^{-1} \exp\left(\frac{pt}{t-1}\right) = \sum_{n=0}^{\infty} L_n(p)\; t^{n}, \quad |t| < 1.$$

Using the less restrictive representation (2.7) for the extended beta function, we can see that (2.9) holds true for $\mathrm{Re}(x) > -1$, y not an integer and $\mathrm{Re}(p) > 0$.

Theorem 1. Let $\mathrm{Re}(\lambda) > -1$, α not an integer and $\mathrm{Re}(p) > 0$, then

$$
\begin{aligned}
(2.10) \qquad D_z^{\alpha,p} z^{\lambda} \log z = {} & \tfrac{z^{\lambda-\alpha} \mathrm{e}^{-2p}}{\Gamma(-\alpha)} \textstyle\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \tfrac{\Gamma(\lambda+m+2)\Gamma(1-\alpha+n)}{\Gamma(3+\lambda-\alpha+m+n)} L_m(p) L_n(p) \\
& \cdot \left[\log z + \Psi(\lambda+m+2) - \Psi(3+\lambda-\alpha+m+n)\right].
\end{aligned}
$$

Proof. Differentiating with respect to λ the left part of (1.6), we get

$$
\begin{aligned}
(2.11) \qquad & \tfrac{\partial}{\partial\lambda} D_z^{\alpha,p} z^{\lambda} = \tfrac{\Gamma(1+\alpha)}{2\pi i} \textstyle\int_0^{(z+)} \tfrac{\partial}{\partial\lambda} \zeta^{\lambda} (\zeta - z)^{-\alpha-1} \exp\left[\tfrac{-pz^2}{\zeta(z-\zeta)}\right] \mathrm{d}\zeta \\
& = \tfrac{\Gamma(1+\alpha)}{2\pi i} \textstyle\int_0^{(z+)} (\log\zeta) \zeta^{\lambda} (\zeta - z)^{-\alpha-1} \exp\left[\tfrac{-pz^2}{\zeta(z-\zeta)}\right] \mathrm{d}\zeta \\
& = D_z^{\alpha,p} z^{\lambda} \log z.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
(2.12) \qquad & D_z^{\alpha,p} z^{\lambda} \log z = \tfrac{\partial}{\partial\lambda} \tfrac{B(\lambda+1,-\alpha;p)}{\Gamma(-\alpha)} z^{\lambda-\alpha} \\
& = \tfrac{\partial}{\partial\lambda} \tfrac{z^{\lambda-\alpha}}{\Gamma(-\alpha)} \mathrm{e}^{-2p} \textstyle\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B(\lambda+m+2, -\alpha+n+1) L_m(p) L_n(p) \\
& = \tfrac{z^{\lambda-\alpha} \mathrm{e}^{-2p}}{\Gamma(-\alpha)} \textstyle\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \tfrac{\Gamma(\lambda+m+2)\Gamma(1-\alpha+n)}{\Gamma(3+\lambda-\alpha+m+n)} L_m(p) L_n(p) \\
& \quad \cdot \left[\log z + \Psi(\lambda+m+2) - \Psi(3+\lambda-\alpha+m+n)\right].
\end{aligned}
$$

We end this section by giving a relation between the extended fractional derivative operator and the classical one.

Theorem 2. Let $f(z)$ be an analytic function satisfying the conditions of existence for the extended fractional derivative (2.2). For α not a negative integer and $\mathrm{Re}(p) > 0$, we have

(2.13)

$$
D_z^{\alpha,p} z^{\lambda} f(z) = e^{-2p} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(p) L_n(p)\, z^{-m-n-2}\, (-\alpha)_{n+1}\, D_z^{\alpha-n-1} z^{\lambda+m+1} f(z).
$$

Proof. We start with the representation (2.2) for the extended fractional derivative operator

$$
(2.14) \qquad D_z^{\alpha,p} z^{\lambda} f(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_0^{(z+)} f(\zeta) \zeta^{\lambda} (\zeta - z)^{-\alpha-1} \exp\left[\frac{-pz^2}{\zeta(z-\zeta)}\right] \mathrm{d}\zeta.
$$

Replacing ζ by uz in (2.14), we obtain

(2.15)

$$
D_z^{\alpha,p} z^{\lambda} f(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_0^{(1+)} f(uz) z^{\lambda-\alpha} u^{\lambda} (u-1)^{-\alpha-1} \exp\left[\frac{-p}{u(1-u)}\right] \mathrm{d}u.
$$

Using the fact that (see [12, eq. 3.5, p. 30])

$$\exp\left[\frac{-p}{t(1-t)}\right] = \mathrm{e}^{-2p}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} L_m(p)L_n(p)\, t^{m+1}(-1)^{n+1}(t-1)^{n+1}, \tag{2.16}$$

we have

$$\begin{aligned} D_z^{\alpha,p} z^{\lambda} f(z) &= \frac{\Gamma(1+\alpha)}{2\pi i}\int_0^{(1+)} f(uz) z^{\lambda-\alpha}\mathrm{e}^{-2p} \\ &\quad \cdot \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} L_m(p)L_n(p)\,(-1)^{n+1}u^{\lambda+m+1}(u-1)^{-\alpha+n}\mathrm{d}u \\ &= \frac{\Gamma(1+\alpha)\,\mathrm{e}^{-2p}}{2\pi i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} L_m(p)L_n(p)\,(-1)^{n+1} \\ &\quad \cdot \int_0^{(1+)} f(uz) z^{\lambda-\alpha} u^{\lambda+m+1}(u-1)^{-\alpha+n}\mathrm{d}u. \end{aligned} \tag{2.17}$$

Now, putting $u = \zeta/z$ in the last equality of (2.17), we get

$$\begin{aligned} D_z^{\alpha,p} z^{\lambda} f(z) = \mathrm{e}^{-2p}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} L_m(p)L_n(p)\, &\frac{(-1)^{n+1}z^{-m-n-2}\,\Gamma(1+\alpha)}{2\pi i} \\ &\cdot \int_0^{(z+)} f(\zeta)\zeta^{\lambda+m+1}(\zeta-z)^{-\alpha+n}\mathrm{d}\zeta. \end{aligned} \tag{2.18}$$

With the help of the representation (2.1) for the classical fractional derivative, we can write

$$D_z^{\alpha,p} z^{\lambda} f(z) = \mathrm{e}^{-2p}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} L_m(p)L_n(p)\,\frac{(-1)^{n+1}z^{-m-n-2}\,\Gamma(1+\alpha)}{\Gamma(\alpha-n)} D_z^{\alpha-n-1} z^{\lambda+m+1} f(z). \tag{2.19}$$

Finally, making use of the well-known property of the gamma function [21, eq. (I.27), p. 240]

$$\Gamma(a-n) = \frac{\Gamma(a)(-1)^n}{(1-a)_n}, \tag{2.20}$$

(2.19) reduces to

$$D_z^{\alpha,p} z^{\lambda} f(z) = \mathrm{e}^{-2p}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} L_m(p)L_n(p)\, z^{-m-n-2}\,(-\alpha)_{n+1}\, D_z^{\alpha-n-1} z^{\lambda+m+1} f(z). \tag{2.21}$$

Remark 2.1. Note that according to (2.4), we also have

$$\begin{aligned} D_{g(z)}^{\alpha,p} g(z)^{\lambda} f(z) = \mathrm{e}^{-2p}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} L_m(p)L_n(p)\, &g(z)^{-m-n-2} \\ &(-\alpha)_{n+1}\, D_{g(z)}^{\alpha-n-1} g(z)^{\lambda+m+1} f(z). \end{aligned} \tag{2.22}$$

3. Leibniz Rule for the Extended Fractional Derivative Operator

In this section, we obtain a generalized Leibniz rule for the extended fractional derivative operator defined on a single loop contour of integration by making use of the method introduced by Osler in [16]. Next, in theorem 4, we give the case where the fractional derivative is calculated with respect of an arbitrary regular and univalent function.

Theorem 3. Let $u(z)$ and $v(z)$ be analytic functions of z on the simply connected region R. Suppose also that 0 is an interior or boundary point of R and that the integral along any simple closed path in R through 0 of $u(z)$, $v(z)$ and $u(z)v(z)$ is zero. Call S the set of all z such that the closed disk $|\zeta - z| \leq |z|$ contains only points ζ in $R \cup \{0\}$. Then

$$D_z^{\alpha,p} z^{\beta+\lambda} u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{n+\gamma} D_z^{\alpha-\gamma-n,p} z^{\beta} u(z) D_z^{\gamma+n} z^{\lambda} v(z) \tag{3.1}$$

for $z \in S$, $\alpha \in C$, $\alpha \neq -1, -2, -3, ...$, $\mathrm{Re}(p) > 0$, $\gamma \in C$, $Re(\beta) > -1$, $\mathrm{Re}(\lambda) > -1$ and $\mathrm{Re}(\beta + \lambda) > -1$.

Proof. By making use of the contours shown in the Figure 2 we know that the extended Cauchy's integral formula for the extended fractional derivatives states that

$$\begin{aligned} D_z^{\alpha,p} z^{\beta+\lambda} u(z)v(z) &= \frac{\Gamma(\alpha+1)}{2\pi i} \int_{C_2} \frac{\xi^{\beta+\lambda} u(\xi)v(\xi)\exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha+1}} d\xi \\ &= \frac{\Gamma(\alpha+1)}{2\pi i} \int_{C_2} \frac{\xi^{\beta} u(\xi)\exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma+1}} \frac{\xi^{\lambda} v(\xi)}{(\xi-z)^{\gamma}} d\xi. \end{aligned} \tag{3.2}$$

Using the elementary Cauchy integral formula we can rewrite (3.2) in the form

$$\begin{aligned} D_z^{\alpha,p} z^{\beta+\lambda} u(z)v(z) &= \frac{\Gamma(\alpha+1)}{-4\pi^2} \int_{C_2} \frac{\xi^{\beta} u(\xi)\exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_3-C_1} \frac{\zeta^{\lambda} v(\zeta)}{(\zeta-z)^{\gamma}(\zeta-\xi)} d\zeta\, d\xi \\ &= \frac{\Gamma(\alpha+1)}{-4\pi^2} \left[\int_{C_2} \frac{\xi^{\beta} u(\xi)\exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_3} \frac{\zeta^{\lambda} v(\zeta)}{(\zeta-z)^{\gamma}(\zeta-\xi)} d\zeta\, d\xi \right. \\ &\quad \left. + \int_{C_2} \frac{\xi^{\beta} u(\xi)\exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_1} \frac{\zeta^{\lambda} v(\zeta)}{(\zeta-z)^{\gamma}(\xi-\zeta)} d\zeta\, d\xi \right]. \end{aligned} \tag{3.3}$$

Replacing C_2 by C_1 in the first term of this last expression and C_2 by C_3 in the second term yield, after elementary manipulation,

$$\begin{aligned} D_z^{\alpha,p} z^{\beta+\lambda} u(z)v(z) &= \frac{\Gamma(\alpha+1)}{-4\pi^2} \left[\int_{C_1} \frac{\xi^{\beta} u(\xi)\exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_3} \frac{\zeta^{\lambda} v(\zeta)}{(\zeta-z)^{\gamma+1}(1-(\xi-z)/(\zeta-z))} d\zeta\, d\xi \right. \\ &\quad \left. + \int_{C_3} \frac{\xi^{\beta} u(\xi)\exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_1} \frac{\zeta^{\lambda} v(\zeta)(\zeta-z)/(\xi-z)}{(\zeta-z)^{\gamma+1}(1-(\zeta-z)/(\xi-z))} d\zeta\, d\xi \right]. \end{aligned} \tag{3.4}$$

Expanding in power series, we obtain

$$
\begin{aligned}
D_z^{\alpha,p} z^{\beta+\lambda} u(z) v(z) = \tfrac{\Gamma(\alpha+1)}{-4\pi^2} \Bigg[& \int_{C_1} \tfrac{\xi^\beta u(\xi) \exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_3} \tfrac{\zeta^\lambda v(\zeta)}{(\zeta-z)^{\gamma+1}} \\
& \left[\textstyle\sum_{n=0}^{N} \left(\tfrac{\xi-z}{\zeta-z}\right)^n + \tfrac{((\xi-z)/(\zeta-z))^{N+1}}{1-(\xi-z)/(\zeta-z)} \right] \mathrm{d}\zeta \, \mathrm{d}\xi \\
& + \int_{C_3} \tfrac{\xi^\beta u(\xi) \exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_1} \tfrac{\zeta^\lambda v(\zeta)}{(\zeta-z)^{\gamma+1}} \\
& \left[\textstyle\sum_{n=1}^{N} \left(\tfrac{\zeta-z}{\xi-z}\right)^n + \tfrac{((\zeta-z)/(\xi-z))^{N+1}}{1-(\zeta-z)/(\xi-z)} \right] \mathrm{d}\zeta \, \mathrm{d}\xi \Bigg] \\
= \textstyle\sum_{n=-N}^{N} & \tfrac{\Gamma(\alpha+1)}{-4\pi^2} \int_{C_2} \tfrac{\xi^\beta u(\xi) \exp\left[\frac{-pz^2}{\xi(z-\xi)}\right] \mathrm{d}\xi}{(\xi-z)^{\alpha-\gamma-n+1}} \int_{C_2} \tfrac{\zeta^\lambda v(\zeta) \mathrm{d}\zeta}{(\zeta-z)^{\gamma+n+1}} \\
- \tfrac{\Gamma(\alpha+1)}{4\pi^2} & \int_{C_1} \tfrac{\xi^\beta u(\xi) \exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma}} \int_{C_3} \tfrac{\zeta^\lambda v(\zeta)}{(\zeta-z)^{\gamma+1}} \tfrac{((\xi-z)/(\zeta-z))^N}{\zeta-\xi} \mathrm{d}\zeta \mathrm{d}\xi \\
- \tfrac{\Gamma(\alpha+1)}{4\pi^2} & \int_{C_3} \tfrac{\xi^\beta u(\xi) \exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_1} \tfrac{\zeta^\lambda v(\zeta)}{(\zeta-z)^{\gamma}} \tfrac{((\zeta-z)/(\xi-z))^N}{\xi-\zeta} \mathrm{d}\zeta \mathrm{d}\xi.
\end{aligned}
\tag{3.5}
$$

Rewriting this last equation in term of the extended fractional derivative operator and of the classical fractional derivative operator, we thus have

$$
\begin{aligned}
D_z^{\alpha,p} z^{\beta+\lambda} u(z) v(z) = & \textstyle\sum_{n=-N}^{N} \binom{\alpha}{n+\gamma} D_z^{\alpha-\gamma-n,p} z^\beta u(z) D_z^{\gamma+n} z^\lambda v(z) \\
& - \tfrac{\Gamma(\alpha+1)}{4\pi^2} \int_{C_1} \tfrac{\xi^\beta u(\xi) \exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma}} \int_{C_3} \tfrac{\zeta^\lambda v(\zeta)}{(\zeta-z)^{\gamma+1}} \tfrac{((\xi-z)/(\zeta-z))^N}{\zeta-\xi} \mathrm{d}\zeta \mathrm{d}\xi \\
& - \tfrac{\Gamma(\alpha+1)}{4\pi^2} \int_{C_3} \tfrac{\xi^\beta u(\xi) \exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma+1}} \int_{C_1} \tfrac{\zeta^\lambda v(\zeta)}{(\zeta-z)^{\gamma}} \tfrac{((\zeta-z)/(\xi-z))^N}{\xi-\zeta} \mathrm{d}\zeta \mathrm{d}\xi.
\end{aligned}
\tag{3.6}
$$

It is easy to see that the remaining two terms vanish as $N \to \infty$ since

$$\left|\frac{\xi - z}{\zeta - z}\right| = \left|\frac{\xi - z}{z}\right| < 1$$

for ζ and ξ not zero in the first remainder, since C_3 is a circle. This is why $z \in S$. A similar assertion holds for the second remainder. The theorem is thus proved.

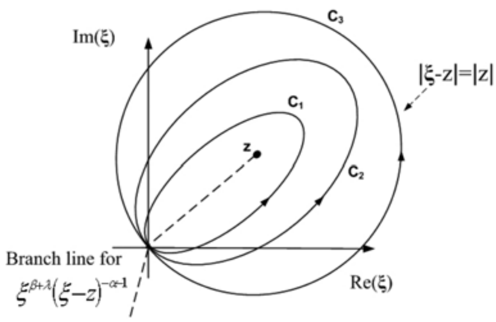

Figure 2. Single loop

Remark 3.1. If in (3.3), we use the Cauchy integral formula to represent the function
$\dfrac{\xi^{\,\beta} u(\xi)\exp\left[\frac{-pz^2}{\xi(z-\xi)}\right]}{(\xi-z)^{\alpha-\gamma+1}}$ instead of $\dfrac{\zeta^{\lambda}v(\zeta)}{(\zeta-z)^{\gamma}}$, we obtain

$$D_z^{\alpha,p} z^{\beta+\lambda} u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{n+\gamma} D_z^{\alpha-\gamma-n} z^{\beta} u(z) D_z^{\gamma+n,p} z^{\lambda} v(z). \tag{3.7}$$

Thus, we can interchange the two operators involved in the left part of (3.1).

Now, considering the fact that we can differentiate fractionally with respect to an arbitrary regular and univalent function $g(z)$, we have the following more general form of Theorem 3.

Theorem 4. With the hypothesis of the previous theorem and the additional conditions: i) $g(z)$ is a regular and univalent function on R. ii) $g^{-1}(0)$ is an interior or boundary point of R. Then
(3.8)

$$D_{g(z)}^{\alpha,p} g(z)^{\beta+\lambda} u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{n+\gamma} D_{g(z)}^{\alpha-\gamma-n,p} g(z)^{\beta} u(z) D_{g(z)}^{\gamma+n} g(z)^{\lambda} v(z)$$

for $z \in g^{-1}(S)$, $\alpha \in C$, $\alpha \neq -1,-2,-3,...$, $\mathrm{Re}(p) > 0$, $\gamma \in C$, $\mathrm{Re}(\beta) > -1$, $\mathrm{Re}(\lambda) > -1$ and $\mathrm{Re}(\beta+\lambda) > -1$.

Remark 3.2. In view of remark (3.1), we have that theorem 4 can be written in the form
(3.9)

$$D_{g(z)}^{\alpha,p} g(z)^{\beta+\lambda} u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{n+\gamma} D_{g(z)}^{\alpha-\gamma-n} g(z)^{\beta} u(z) D_{g(z)}^{\gamma+n,p} g(z)^{\lambda} v(z).$$

4. Applications

In this section, we examine some interesting special cases which can be obtained from theorem 4.

Example 4.1. If we set $g(z) = z$, $u(z) = z^{-A-1}$, $v(z) = z^{-B-1}$, $\beta = D$, $\lambda = C$, $\alpha = C-A-1$ and $\gamma = C-1$, then for $C-A-1$ not a negative integer and $Re(C+D-A-B) > 1$, we obtain

$$\frac{B(D+C-A-B-1,1+A-C;p)}{\Gamma(1+A-C)\Gamma(C-A)\Gamma(C-B)} = \sum_{n=-\infty}^{\infty} \frac{B(D-A,A+n;p)}{\Gamma(C+n)\Gamma(A+n)\Gamma(1-A-n)\Gamma(1-B-n)}. \tag{4.1}$$

Putting $p = 0$ and the reflection formula for the gamma function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{4.2}$$

we get the well-known Dougall's formula [9]

$$\frac{\pi^2\Gamma(C+D-A-B-1)}{\Gamma(C-A)\Gamma(C-B)\Gamma(D-A)\Gamma(D-B)\sin(\pi A)\sin(\pi B)} = \sum_{n=-\infty}^{\infty}\frac{\Gamma(A+n)\Gamma(B+n)}{\Gamma(C+n)\Gamma(D+n)} \tag{4.3}$$

which holds for $\mathrm{Re}(C+D-A-B)>1$.

Example 4.2. Let $g(z)=z$, $u(z)=(1-z)^{-a}$, $v(z)=(1-z)^{-A}$ and set $\beta=b-1$, $\lambda=B-1$, $\alpha=b+B-d-D$ and $\gamma=B-D$ in the new generalized Leibniz rule. For $\mathrm{Re}(b)>0$, $\mathrm{Re}(B)>0$, $\mathrm{Re}(b+B)>1$ and $\mathrm{Re}(z)<1/2$ we have

$$\frac{\Gamma(B+b-1)}{\Gamma(D+d-1)\Gamma(b)\Gamma(B)\Gamma(B+b-D-d+1)}\;F_p\left[a+A,B+b-1;D+d-1;z\right]$$

$$= \sum_{n=-\infty}^{\infty}\frac{F_p\left[a,b;d+n;z\right]\;F\left[A,B;D-n;z\right]}{\Gamma(d+n)\Gamma(D-n)\Gamma(1+B-D+n)\Gamma(1+b-d-n)}. \tag{4.4}$$

Note that the second hypergeometric function appearing in the bilateral sum is the classical one and that we have used the identity

$$D_z^{\lambda-\alpha,p}z^{\lambda-1}(1-z)^{-\mu}=\frac{\Gamma(\lambda)}{\Gamma(\alpha)}z^{\alpha-1}F_p(\mu,\lambda;\alpha;z).$$

Here again, if $p=0$, we recover a result obtained by Osler in [16, eq. 9, p. 670], namely,

$$\frac{\Gamma(B+b-1)}{\Gamma(D+d-1)\Gamma(b)\Gamma(B)\Gamma(B+b-D-d+1)}\;F\left[a+A,B+b-1;D+d-1;z\right]$$

$$= \sum_{n=-\infty}^{\infty}\frac{F\left[a,b;d+n;z\right]\;F\left[A,B;D-n;z\right]}{\Gamma(d+n)\Gamma(D-n)\Gamma(1+B-D+n)\Gamma(1+b-d-n)} \tag{4.5}$$

where all the hypergeometric functions involved are the classical ones.

Example 4.3. In example 2.1, we used table 1 to define the extended Legendre function of the first kind by

$$P_{\mu,p}(z)=\frac{1}{\Gamma(1+\mu)2^{\mu}}D_{1-z}^{\mu,\,p}(1-z^2)^{\mu}. \tag{4.6}$$

Since we can extend special functions that possess a representation in terms of classical fractional derivative by this way, let the extended associated Legendre function of the first kind be defined by

$$P_{\nu,p}^{\mu}(z)=\frac{(1-z^2)^{\mu/2}}{2^{\nu}\Gamma(1+\nu)}D_{1-z}^{\mu+\nu,\,p}(1-z^2)^{\nu}. \tag{4.7}$$

Now, put $g(z) = 1 - z$, $u(z) = 1$, $v(z) = (1+z)^\nu$, $\alpha = \nu + \mu$, $\beta = 0$, $\lambda = \nu$, theorem 4 becomes

(4.8)
$$D_{1-z}^{\nu+\mu,p}(1-z)^\nu(1+z)^\nu = \sum_{n=-\infty}^{\infty} \binom{\nu+\mu}{n+\gamma} D_{1-z}^{\nu+\mu-\gamma-n,p}(1-z)^0 D_{1-z}^{n+\gamma}(1-z)^\nu(1+z)^\nu.$$

Multiplying both sides by $\dfrac{(1-z^2)^{\mu/2}}{2^\nu\Gamma(1+\nu)}$, we can write (4.8) as

(4.9)
$$\begin{aligned} P_{\nu,p}^{\mu}(z) &= \tfrac{(1-z^2)^{\mu/2}}{2^\nu\Gamma(1+\nu)} \textstyle\sum_{n=-\infty}^{\infty} \binom{\nu+\mu}{n+\gamma} D_{1-z}^{\nu+\mu-\gamma-n,p}(1-z)^0 D_{1-z}^{n+\gamma}(1-z)^\nu(1+z)^\nu \\ &= \tfrac{(1-z^2)^{\mu/2}}{2^\nu\Gamma(1+\nu)} \textstyle\sum_{n=-\infty}^{\infty} \frac{\Gamma(1+\mu+\nu)B(1,-\mu-\nu+\gamma+n;\,p)\,(1-z)^{-\mu-\nu+\gamma+n}}{\Gamma(1+\gamma+n)\Gamma(1+\mu+\nu-\gamma-n)\Gamma(-\mu-\nu+\gamma+n)} D_{1-z}^{n+\gamma}(1-z^2)^\nu. \end{aligned}$$

Observing that

$$D_{1-z}^{\gamma+n}(1-z^2)^\nu = P_\nu^{\gamma-\nu+n}(z)\ \frac{2^\nu\Gamma(1+\nu)}{(1-z^2)^{(\gamma-\nu+n)/2}} \tag{4.10}$$

and substituting in (4.9), we finally obtain

(4.11)
$$\begin{aligned} P_{\nu,p}^{\mu}(z) &= \tfrac{(1-z^2)^{\mu/2}}{2^\nu\Gamma(1+\nu)} \textstyle\sum_{n=-\infty}^{\infty} \frac{\Gamma(1+\mu+\nu)B(1,-\mu-\nu+\gamma+n;\,p)\,(1-z)^{-\mu-\nu+\gamma+n}}{\Gamma(1+\gamma+n)\Gamma(1+\mu+\nu-\gamma-n)\Gamma(-\mu-\nu+\gamma+n)} \\ &\cdot P_\nu^{\gamma-\nu+n}(z)\ \tfrac{2^\nu\Gamma(1+\nu)}{(1-z^2)^{(\gamma-\nu+n)/2}} \\ &= \left(\tfrac{1-z}{1+z}\right)^{(\gamma-\mu-\nu)/2} \textstyle\sum_{n=-\infty}^{\infty} \binom{\nu+\mu}{n+\gamma} \frac{B(1,-\mu-\nu+\gamma+n;\,p)}{\Gamma(-\mu-\nu+\gamma+n)} P_\nu^{\gamma-\nu+n}(z)\ \left(\tfrac{1-z}{1+z}\right)^{n/2}. \end{aligned}$$

The latter result holds true for $Re(\nu) > -1$ and $Re(z) > 0$. If $p = 0$, we find a result obtained by Osler [16, eq. 16, p. 671].

Example 4.4. Let $g(z) = z^2$, $u(z) = 1$, $v(z) = \frac{\cos z}{z}$, set $\beta = 0$, $\lambda = 0$, $\alpha = -\nu - 1/2$ and $\gamma = -1/2 - b$ in theorem 4 and let the extended Bessel function of the first kind be defined by

$$J_{\nu,p}(z) = \frac{z^{-\nu}}{2^\nu\Gamma(1/2)} D_{z^2}^{-\nu-1/2,p}\ \frac{\cos z}{z}. \tag{4.12}$$

Then, we have

$$\begin{aligned} J_{\nu,p}(z) &= \tfrac{z^\nu}{2^\nu\Gamma(1/2)\Gamma(1/2+\nu)} \textstyle\sum_{k=0}^{\infty} \frac{B(k+1/2,\nu+1/2;\,p)}{(1/2)_k k!} \left(\frac{-z^2}{4}\right)^k \\ &= \textstyle\sum_{n=-\infty}^{\infty} \binom{-\nu-1/2}{n-1/2-b} \frac{B(1,\nu-b+n;\,p)}{\Gamma(\nu-b+n)} \left(\frac{z}{2}\right)^{\nu-b+n} J_{b-n}(z). \end{aligned} \tag{4.13}$$

Specially, if $p = 0$, then we get a result mentioned by Osler in [16, eq. 12, p. 671].

5. Conclusion

The usefulness of generalized Leibniz rule to obtain new series expansion is a well known fact. We, thus, have presented a new Leibniz rule for the extended fractional derivative operator based upon the Cauchy integral formula. We gave some interesting special cases. We also have expressed the extended fractional derivative operator in terms of the classical one. A large number of applications of this Leibniz rule are under investigation and will appear soon.

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