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Volterra integral equation method for solving some hyperbolic equation problems ⋆

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Summary. The Cauchy problem for a hyperbolic equation with function coefficients of the first partial derivatives with respect to time and space variables is considered. It is proved by Sobolev's method that solution of this problem satisfies a 3-D Volterra integral equation. Using this fact the uniqueness theorem for an inverse problem is proved.

Key words: hyperbolic equation of the second order, Cauchy problem, Volterra integral equation, inverse problem.

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1. Introduction

The theory of linear hyperbolic equations with function coefficients is very well developed. There are general existence and uniqueness theorems for weak and classical solutions of initial value and initial boundary value problems (see, for example [1–3]). Some particular cases of hyperbolic equations have interesting properties which are useful for numerical methods, inverse problems theory and others. For example, the solution of the Cauchy problem for the wave equation with three space variables and the constant velocity coefficient is

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given by Kirchhoff's explicit formula. If the speed coefficient is a function then there is no explicit formula for the solution. However if the speed coefficient depending on three space variables is a smooth function then the solution of the Cauchy problem for the wave equation satisfies a 3-D Volterra integral equation. This result was obtained by S.Sobolev [4] and is a generalization of Kirchhoff's formula. This Sobolev result was generalized for some hyperbolic equations [5,6].

The first part of the present paper is related to generalization of Sobolev's result for other cases of hyperbolic equations. More precisely we generalized Sobolev Volterra integral equation method of hyperbolic equations with function coefficients of the first partial derivatives with respect to time and space variables. This property of the Cauchy problem solution (that is, to satisfy a 3-D Volterra integral equation), we apply to prove the uniqueness theorem for an inverse problem in the second part of this paper.

2. Initial value problem for hyperbolic equation and property of its solution

Let us consider the following scalar hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta_x u + \sum_{j=1}^{3} b_j(x) \frac{\partial u}{\partial x_j} + q(x) \frac{\partial u}{\partial t} + f(x,t), \tag{1}$$

$$x \in \mathbb{R}^3,\ t > 0,$$

subject to the initial data

$$u(x,0) = g(x), \quad \frac{\partial u}{\partial t}(x,0) = h(x), \tag{2}$$

where

$$\Delta_x = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$$

is the Laplace operator. We assume that c is a positive constant, $b_j(x)$, $q(x) \in C^2(\mathbb{R}^3), j = 1,2,3,\ g(x) \in C^2(\mathbb{R}^3) \cap H^4(\mathbb{R}^3),\ h(x) \in C^1(\mathbb{R}^3) \cap H^3(\mathbb{R}^3)$, $\left(\frac{\partial}{\partial t}\right)^j f(x,t) \in C\Big([0,T]; H^{4-j}(\mathbb{R}^3)\Big)$, $j = 0,1,2; H^m(\mathbb{R}^3)$ $(m = 1,2,3,4)$ are the Sobolev spaces. The theory of hyperbolic equations [3] contains existence and uniqueness theorems of a solution

$u(x,t)$ of (1), (2) satisfying

$$\frac{\partial^j u}{\partial t^j}(x,t) \in C\left([0,T];\, H^{4-j} \in (\mathbb{R}^3) \cap C^{2-j}(\mathbb{R}^3)\right),\ j = 0,1,2,$$

$$\frac{\partial^3}{\partial t^3} \in C\left([0,T]; H^1 \in (\mathbb{R}^3)\right).$$

The main goal of this section is to study a property of this solution. This property means that solution of (1), (2) satisfies a 3-D Volterra integral equation. To get this property we use Sobolev's method [4, 5]. Let $u(x,t)$ be solution of (1), (2). Consider other function $u_1(x,t)$ which is given by

$$u_1(x,t) = u\left(x, t - \frac{|x-x^0|}{c}\right), \tag{3}$$

where

$$x^0 = \left(x_1^0, x_2^0, x_3^0\right) \in \mathbb{R}^3$$

is a parameter. Consider the differential operator L which is defined by the formula

$$L_x u_1 \equiv \left(\Delta_x u_1 + \frac{1}{c^2}\sum_{j=1}^{3} b_j(x)\frac{\partial u_1}{\partial x_j}\right).$$

The following relation holds

$$\begin{aligned}\sigma L_x u_1 = {} & -2\sigma \nabla_x \tau \nabla_x \tfrac{\partial u_1}{\partial t} - \tfrac{1}{c^2}\sigma f(x, t-\tau(x)) \\ & - \sigma \tfrac{\partial u_1}{\partial t}\left(\Delta_x \tau(x) + \tfrac{1}{c^2} q(x) + \tfrac{1}{c^2}\textstyle\sum_{j=1}^{3} b_j(x)\tfrac{\partial \tau(x)}{\partial x_j}\right),\end{aligned} \tag{4}$$

for any function $\sigma(x)$ from $C^2(\mathbb{R}^3)$ and $\tau(x) = |x-x^0|/c$. Here ∇_x is the gradient operator.

Let $\sigma = \sigma(x,x^0)$ be solution of the following problem

$$2\nabla_x \tau(x) \nabla_x \sigma + \sigma\left(\triangle_x \tau(x) - \frac{1}{c^2}q(x) - \frac{1}{c^2}\sum_{j=1}^{3} b_j(x)\frac{\partial \tau}{\partial x_j}\right) = 0, \tag{5}$$

$$\sigma(x,x^0) = O\left(\frac{1}{|x-x^0|}\right) \qquad \text{as} \qquad x \longrightarrow x^0. \tag{6}$$

Then the equality (4) may be written as follows

$$\sigma L_x u_1 = div_x \left(-\frac{\partial u_1}{\partial t} 2\sigma \nabla_x \tau \right) - \frac{1}{c^2} \sigma f(x, t - \tau(x)). \tag{7}$$

Remark 1. We note that we can find a solution of (5), (6) using the method of characteristics. This solution is given by the explicit formula

$$\sigma(x, x^0) = \frac{1}{|x - x^0|} \exp \left(\frac{|x - x^0|}{2c} \int_0^1 q(x^0 + (x - x^0)z) dz \right) \tag{8}$$
$$\times \exp \left(\frac{1}{2c^2} \int_0^1 \sum_{j=1}^3 b_j(x^0 + (x - x^0)z)(x_j - x_j^0) dz \right).$$

Using the formula (8) we obtain the following properties of the function $\sigma(x, x^0)$:

1. $\lim_{x \to x^0} \sigma(x, x^0)|x - x^0| = 1;$
2. $\sigma(x, x^0)$ is twice continuously differentiable function if $x \neq x^0$;
3. $|L_x^* \sigma(x, x^0)| \leq O\left(|x - x^0|^{-2}\right), \quad$ as $\quad x \to x^0$;
4. $\lim_{r \to +0} \int\int_{|x - x^0| = r} \frac{\partial \sigma(x, x^0)}{\partial n} dS = -4\pi.$

Here

$$L_x^* \sigma(x, x^0) = \Delta_x \sigma(x, x^0) - \frac{1}{c^2} \sum_{j=1}^3 \frac{\partial}{\partial x_j} \left(b_j(x) \sigma(x, x^0) \right)$$

is the adjoint operator to L_x.

Let $u_1(x, t)$, $\sigma(x, x^0)$ be functions defined by (3), (8). Using (7) and Green's formula

$$\int\int\int_{|x - x^0| \leq ct} \left(\sigma(x, x^0) L_x u_1(x, t) - u_1(x, t) L_x^* \sigma(x, x^0) \right) dx$$
$$= \int\int_{|x - x^0| = ct} \Bigg(\sigma(x, x^0) \frac{\partial u_1(x,t)}{\partial n} - u_1(x, t) \frac{\partial \sigma(x,x^0)}{\partial n}$$
$$+ \frac{1}{c^2} \sum_{j=1}^3 b_j(x) n_j \sigma(x, x^0) u_1(x, t) \Bigg) dS_x,$$

we find

$$(9)\quad \int\int\int_{|x-x^0|\le ct} div_x\left(-\frac{\partial u_1(x,t)}{\partial t}2\sigma(x,x^0)\nabla_x\left(\frac{|x-x^0|}{c}\right)\right)dx$$
$$-\ \tfrac{1}{c^2}\int\int\int_{|x-x^0|\le ct}\sigma(x,x^0)f\left(x,t-\tfrac{|x-x^0|}{c}\right))dx$$
$$-\ \int\int\int_{|x-x^0|\le ct}u_1(x,t)L_x^*\sigma(x,x^0)dx$$
$$=\int\int_{|x-x^0|=ct}\left(\sigma(x,x^0)\tfrac{\partial u_1(x,t)}{\partial n}-u_1(x,t)\tfrac{\partial\sigma(x,x^0)}{\partial n}\right.$$
$$\left.+\ \tfrac{1}{c^2}\sum_{j=1}^3 b_j(x)n_j\sigma(x,x^0)u_1(x,t)\right)dS_x.$$

Applying Ostrogradskii's formula the equation (9) may be written as

$$(10)\qquad \frac{1}{c^2}\int\int\int_{|x-x^0|\le ct}\sigma(x,x^0)f\left(x,t-\frac{|x-x_0|}{c}\right)dx$$
$$+\ \int\int\int_{|x-x^0|\le ct}u_1(x,t)L_x^*\sigma(x,x^0)dx$$
$$+\ \int\int_{|x-x^0|=ct}\left(\sigma(x,x^0)\tfrac{\partial u_1(x,t)}{\partial n}-u_1(x,t)\tfrac{\partial\sigma(x,x^0)}{\partial n}\right)dS_x$$
$$+\ \tfrac{1}{c^2}\int\int_{|x-x^0|=ct}\sum_{j=1}^3 b_j(x)n_j\sigma(x,x^0)u_1(x,t)dS_x$$
$$+\ \tfrac{1}{c}\int\int_{|x-x^0|=ct}2\sigma\tfrac{\partial u_1(x,t)}{\partial t}dS=0.$$

Using the properties 1)-4) of $\sigma(x,x^0)$, (see Remark 1) and the relations

$$u_1\left(x,\frac{|x-x^0|}{c}\right)=u(x,0)=g(x),$$
$$\frac{\partial u_1(x,t)}{\partial t}\Big|_{t=|x-x^0|c^{-1}}=\frac{\partial}{\partial t}u(x,0)=h(x)$$

the formula (10) can be written as

$$(11)\qquad u(x,t)=F(x,t)$$
$$+\ \tfrac{1}{4\pi}\int\int\int_{|x-\xi|\le ct}u\left(\xi,t-\tfrac{|\xi-x|}{c}\right)L_\xi^*\sigma(\xi,x)d\xi,$$

where

$$F(x,t) = \frac{1}{4\pi}\int\int_{|x-\xi|=ct}\Bigg\{\sigma(\xi,x)\frac{\partial g(\xi)}{\partial n} - g(\xi)\frac{\partial \sigma(\xi,x)}{\partial n}$$

$$+ \left(\frac{1}{c^2}\sum_{j=1}^{3} b_j(\xi)n_j\right)\sigma(\xi,x)g(\xi) + \frac{\sigma(\xi,x)}{c}h(\xi)\Bigg\}dS_\xi$$

$$+ \frac{1}{4\pi c^2}\int\int\int_{|x-\xi|\le ct}\sigma(\xi,x)f\left(\xi,t-\frac{|\xi-x|}{c}\right)d\xi.$$

Remark 2. The formula (10) for a particular case $b_j \equiv 0$, $j = 1,2,3;$ $q(x) = 0$ will be written as Kirchhoff's formula for the solution of the Cauchy problem for the wave equation.

Remark 3. Let us consider the function in the form of Neumann's series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

where $u_0(x,t) = F(x,t)$, $n \ge 1$,

$$u_n(x,t) = \frac{1}{4\pi}\int\int\int_{|\xi-x|\le ct} u_{n-1}\left(\xi, t-\frac{|\xi-x|}{c}\right)L_\xi^*\sigma(\xi,x)d\xi.$$

We can show that

1) the series $\sum_{n=0}^{\infty} u_n(x,t)$ is uniformly convergent to a continuous function $u(x,t)$ on $\triangle(T)$,

2) the function $u(x,t)$ is a unique solution of the integral equation (10) for $(x,t) \in \triangle(T)$,

where

$$\triangle(T) = \{(x,t) : 0 \le t \le T - \frac{|x|}{c}\},\ T > 0.$$

3. Uniqueness theorem of an inverse problem

In this section the property of the Cauchy problem solution, which was studied in the section 2, is applied to prove the uniqueness theorem of an inverse problem for the integral equation (1). We suppose here that q is a even twice continuously differentiable function depending on x_3, c is a constant, $b_j(x)$, $j = 1,2$, $g(x)$, $h(x)$, $f(x,t)$ are even and b_3 is odd functions satisfying conditions of the section 2.

The main object of the study here is the following inverse problem.
Inverse problem. Let T be a positive number, $X = [-T/c, T/c]$, $f(x,t)$, $g(x)$, $h(x)$ be given functions, $q(x_3) \in C^2(X)$ be unknown even function. Find $q(x_3)$ if the solution of (1), (2) complies with the data

$$u(0,t) = G(t), \tag{12}$$

where $G(t)$ is a function known for $t \in [0,T]$.

Remark 4. The similar inverse problems for Klein-Gordon-Fock equation were studied by V.G.Romanov [6], Rakesh [7].

The main result of this section is the uniqueness theorem.

Theorem. Let $h(0,0,x_3) \neq 0$ for $x_3 \in X$ and $q_i(x_3)$, $i = 1,2$ be solutions of the inverse problem corresponding the same data $G(t)$, $t \in [0,T]$. Then $q_1(x_3) \equiv q_2(x_3)$ for $x_3 \in X$.

Proof. Let $u_i(x,t)$, $i = 1,2$ be two solutions of the Cauchy problem (1), (2) corresponding to $q = q_i(x_3)$, $i = 1,2$, respectively. Subtracting equations (1)-(3) for $q = q_2(x_3)$ from equations (1)-(3) for $q = q_1(x_3)$ we find

$$\frac{\partial^2 \tilde{u}}{\partial t^2} = c^2 \triangle \tilde{u} + \sum_{j=1}^{3} b_j(x) \frac{\partial \tilde{u}}{\partial x_j} + q_1(x_3) \frac{\partial \tilde{u}}{\partial t} + \tilde{q}(x_3) \frac{\partial u_2(x,t)}{\partial t}, \tag{13}$$

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3,\ t > 0,$$

$$\tilde{u}(x,0) = 0, \quad \frac{\partial \tilde{u}(x,0)}{\partial t} = 0, \quad x \in \mathbb{R}^3, \tag{14}$$

$$\tilde{u}(0,t) = 0, \tag{15}$$

where

$$\tilde{q}(x_3) = q_1(x_3) - q_2(x_3), \quad \tilde{u}(x,t) = u_1(x,t) - u_2(x,t).$$

The Cauchy problem (13), (14) is similar to (1), (2). Using reasoning of the section 2, we find that the solution $\tilde{u}(x,t)$ of the problem (13), (14) satisfies the following integral equation which is similar to the equation (11).

$$\begin{aligned} \tilde{u}(x,t) &= \tfrac{1}{4\pi c^2} \int\int\int_{|x-\xi| \le ct} \sigma(\xi,x) \tilde{q}(\xi_3) \tfrac{\partial u_2}{\partial t} \left(\xi, t - \tfrac{|\xi - x|}{c} \right) d\xi \\ &+ \tfrac{1}{4\pi} \int\int\int_{|x-\xi| \le ct} \tilde{u} \left(\xi, t - \tfrac{|\xi - x|}{c} \right) L^*_\xi \sigma(\xi,x) d\xi, \end{aligned} \tag{16}$$

where $\sigma(x, x^0)$ is defined by the formula (8) in which $q = q_1(x_3)$.

Using spherical coordinates

$$\xi = x + r\alpha, \ \ \alpha = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta),$$

$$0 \le \varphi < 2\pi, \ \ 0 < \theta < \pi, \ \ dw = \sin\theta d\theta d\varphi,$$

the second integral of (16) can be written in the form

$$(17) \qquad \frac{1}{4\pi} \int_0^{ct} \int_0^{2\pi} \int_0^{\pi} \tilde{u}\left(\xi, t - \frac{|\xi - x|}{c}\right) L_\xi^* \sigma(\xi, x)\Big|_{\xi = x + r\alpha} r^2 dw dr.$$

The first integral of (16) may be written in terms of coordinates (r, φ, ξ_3). These coordinates we get if θ in the spherical coordinates is changed by ξ_3 according to the formula $\xi_3 = x_3 + r\cos\theta$. As a result we have

$$\bar{\xi} = \bar{x} + \sqrt{r^2 - (\xi_3 - x_3)^2}\nu, \ \bar{\xi} = (\xi_1, \xi_2), \ \bar{x} = (x_1, x_2),$$

$$\nu = (\cos\varphi, \sin\varphi), \ 0 \le \varphi < 2\pi, \ d\xi = -r d\varphi d\xi_3 dr,$$

and the first integral of (16) is presented in the form

$$(18) \qquad \frac{1}{4\pi c^2} \int_0^{ct} \int_0^{2\pi} \int_{x_3 - r}^{x_3 + r} r \Bigg[\sigma(\xi, x)\tilde{q}(\xi_3) \times \frac{\partial u_2}{\partial t}\left(\xi, t - \frac{r}{c}\right)\Bigg]\Bigg|_{\bar{\xi} = \bar{x} + \sqrt{r^2 - (\xi_3 - x_3)^2}\nu} d\xi_3 d\varphi dr.$$

Applying the operator $\left(\frac{\partial}{\partial t}\right)^2$ to (16) and using presentations (17), (18) for integral of (16) we find

$$(19) \qquad \frac{\partial^2 \tilde{u}}{\partial t^2}(x, t) = \frac{c}{2}\Bigg\{\Big(t\sigma(\xi, x)h(\xi)\Big)\Big|_{\bar{\xi} = \bar{x}, r = ct} \tilde{q}(\xi_3)\Bigg\}\Bigg|_{\xi_3 = x_3 - ct}^{\xi_3 = x_3 + ct}$$

$$+ \frac{1}{4\pi} \int_0^{2\pi} \int_{x_3 - ct}^{x_3 + ct} \Bigg\{\frac{\partial}{\partial t}\Bigg[\big(t\sigma(\xi, x)h(\xi)\big)\Big|_{\bar{\xi} = \bar{x} + \sqrt{(ct)^2 - (\xi_3 - x_3)^2}\nu, r = ct}\Bigg]$$

$$+ \left(t\sigma(\xi, x)\frac{\partial^2 u_2}{\partial t^2}(\xi, 0)\right)\Bigg|_{\bar{\xi} = \bar{x} + \sqrt{(ct)^2 - (\xi_3 - x_3)^2}\,\nu, r = ct}\Bigg\}\tilde{q}(\xi_3) d\xi_3 d\varphi$$

$$+\frac{1}{4\pi c^2}\int_0^{ct}\int_0^{2\pi}\int_{x_3-r}^{x_3+r}\Bigg[r\sigma(\xi,x)$$

$$\times\frac{\partial^3 u_2}{\partial t^3}\left(\xi,t-\frac{r}{c}\right)\Bigg]\Bigg|_{\bar\xi=\bar x+\sqrt{r^2-(\xi_3-x_3)^2}\ \nu}\tilde q(\xi_3)d\xi_3 d\varphi dr$$

$$+\frac{1}{4\pi}\int_0^{ct}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\partial^2\tilde u}{\partial t^2}\left(\xi,t-\frac{r}{c}\right)L_\xi^*\sigma(\xi,x)\right]\Bigg|_{\xi=x+r\alpha}r^2 dw dr.$$

Substituting $x = 0$ into (19) and using formulas (12), (8), the evenness of $h, f, g, q_1, q_2, \tilde q, b_j$, $j = 1, 2$ and oddness of b_3 with respect to x_3 we obtain

$$\tilde q(ct)=\frac{1}{c}\left[\left(t\sigma(\xi,0)h(\xi)\right)\Big|_{\bar\xi=0,\,\xi_3=ct}\right]^{-1}\Bigg\{G(t)$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\int_0^{ct}\Bigg[\frac{\partial}{\partial t}\left(t\sigma(\xi,0)h(\xi)\Big|_{\bar\xi=\sqrt{(ct)^2-\xi_3^2}\ \nu,\,r=ct}\right)$$

$$+t\sigma(\xi,0)\frac{\partial^2 u_2}{\partial t^2}(\xi,0)\Big|_{\bar\xi=\sqrt{(ct)^2-\xi_3^2}\nu,\,r=ct}\Bigg]\tilde q(\xi_3)d\xi_3 d\varphi$$

$$-\frac{1}{4\pi c^2}\int_0^{ct}\int_0^{2\pi}\int_{-r}^{r}r\sigma(\xi,0)\frac{\partial^3 u_2}{\partial t^3}\left(\xi,t-\frac{r}{c}\right)\Bigg|_{\bar\xi=\sqrt{r^2-\xi_3^2}\ \nu}\tilde q(\xi_3)d\xi_3 d\varphi dr$$

$$-\frac{1}{4\pi}\int_0^{ct}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\partial^2\tilde u}{\partial t^2}\left(\xi,t-\frac{r}{c}\right)L_\xi^*\sigma(\xi,0)\right]\Bigg|_{\xi=r\alpha}r^2 dw dr\Bigg\}. \tag{20}$$

Equations (19), (20) may be written as a system of two integral equations relative to two functions $W(x,t)$, $\tilde q(ct)$, where

$$W(x,t)=\frac{\partial^2\tilde u}{\partial t^2}(x,t)-\left\{\frac{c}{2}\left[t\sigma(\xi,x)h(\xi)\right]\Big|_{\bar\xi=\bar x,\,r=ct}\tilde q(\xi_3)\right\}\Bigg|_{\xi_3=x_3-ct}^{\xi_3=x_3+ct}.$$

This system will be homogeneous Volterra type with the polar kernel. According to the theory of Volterra integral equations this system has zero solution only. Therefore $\tilde q(x_3) \equiv 0$ for $x_3 \in [0, X]$, and the theorem is proved.

Remark 5. The property of the Cauchy problem solution (that is to satisfy a 3-D Volterra integral equation) can be generalized for general hyperbolic equations of the second order with the function coefficients depending on space variables. The uniqueness theorems for inverse problems of these general hyperbolic equations may be proved using this property also.

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