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α-PREIRRESOLUTE FUNCTIONS AND β-PREIRRSOLUTE FUNCTIONS

Abstract. The purpose of this paper is to introduce new two classes of functions called α-preirresolute functions and β-preirrsolute functions in topological spaces. Some properties and several characterizations of these types of functions are obtained. Also, we investigate the relationship between these clasess of functions and other classes of non-continuous functions.

1. Introduction

Recall that four classes of functions called strongly M-precontinuity [3], strongly α-irresoluteness [9], α-irresoluteness [10] and preirrsoluteness [16]. In 2000, Beceren [4] introduced the notion of almost α-irresoluteness. Recently, Beceren and Noiri [5] introduced the concept of α-precontinuity.

The purpose of the present paper is to introduce and investigate notions of new classes of functions, namely α-preirresolute functions and β-preirresolute functions and give several characterizations and their properties. Relations between these types of functions and other classes of functions are obtained. The new class of α-preirresolute functions, which is stronger than preirresolute functions [16], is a generalization of strongly M-precontinuous functions [3]. The new class of β-preirresolute functions, which is stronger than almost α-irresolute functions [4], is a generalization of preirresolute functions [16].

2. Preliminaries

Throughout this note, spaces always mean topological spaces and $f : X \to Y$ denotes a single valued function of a space X into a space Y. Let

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S be a subset of space X. The closure and the interior of S are denoted by $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$, respectively.

DEFINITION 2.1. A subset S of space X is said to be *α-open* [15] (resp. *preopen* [11], *β-open* [1]) if $S \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S)))$ (resp. $S \subset \mathrm{Int}(\mathrm{Cl}(S))$, $S \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(S)))$).

The family of all α-open (resp. preopen, β-open) sets in a space X is denoted by τ^α (resp.$\mathrm{PO}(X)$, $\beta\mathrm{O}(X)$). It is shown in [15] that τ^α is a topology for X. Moreover, $\tau \subset \tau^\alpha \subset \mathrm{PO}(X) \subset \beta\mathrm{O}(X)$. The complement of an α-open (resp. preopen, β-open) set is said to be *α-closed* [13] (resp. *preclosed* [11], *β-closed* [1]). The intersection of all preclosed sets containing a subset S is called the *preclosure* [12] of S and is denoted by $\mathrm{pCl}(S)$; the union of all preopen sets contained in S is called the *preinterior* [12] of S and is denoted by $\mathrm{pInt}(S)$.

DEFINITION 2.2. A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be *strongly M-precontinuous* [3] (resp. *preirresolute* [16]) if $f^{-1}(V)$ is open (resp. preopen) in X for every preopen set V of Y.

DEFINITION 2.3. A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be *strongly α-irresolute* [9] (resp. *α-irresolute* [10], *α-precontinuous* [5], *almost α-irresolute* [4]) if $f^{-1}(V)$ is open (resp. α-open, preopen, β-open) in X for every α-open set V of Y.

DEFINITION 2.4. A function $f : (X, \tau) \to (Y, \upsilon)$ is said to be *α-preirresolute* (resp. *β-preirresolute*) if $f^{-1}(V)$ is α-open (resp. β-open) in X for every preopen set V of Y.

From the definitions stated above, we obtain the following diagram:

strongly M-percontinuity	→	α-preirresoluteness	→	preirresoluteness	→	β-preirresoluteness
↓		↓		↓		↓
strongly α-irresoluteness	→	α-irresoluteness	→	α-percontinuity	→	almost $\alpha-$irresoluteness

REMARK 2.1. However, converses of the above implications are not true, in general, by [5, Examples 2.1 and 2.2] and the following examples.

EXAMPLE 2.1. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be a function defined as follows: $f(a) = f(b) = a$ and $f(c) = b$. Then f is α-preirresolute, but it is not strongly α-irresolute.

EXAMPLE 2.2. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a, b, c\}\}$ and $\upsilon = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. Let $f : (X, \tau) \to (X, \upsilon)$ be a function defined as follows: $f(a) = f(b) = f(c) = b$ and $f(d) = c$. Then f is strongly α-irresolute, but it is not β-preirresolute.

3. α-preirresolute functions

THEOREM 3.1. *For a function $f : (X, \tau) \to (Y, \upsilon)$, the following are equivalent:*

(a) *f is α-preirresolute;*

(b) *$f : (X, \tau^\alpha) \to (Y, \upsilon)$ is strongly M-percontinuous;*

(c) *For each $x \in X$ and each preopen set V of Y containing $f(x)$, there exists an α-open set U of X containing x such that $f(U) \subset V$;*

(d) *$f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))))$ for every preopen set V of Y;*

(e) *$f^{-1}(F)$ is α-closed in X for every preclosed set F of Y;*

(f) *$\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(B)))) \subset f^{-1}(\mathrm{pCl}(B))$ for every subset B of Y;*

(g) *$f(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))) \subset \mathrm{pCl}(f(A))$ for every subset A of X.*

Proof. (a)$\Rightarrow$(b). Let $x \in X$ and let V be any preopen set of Y containing $f(x)$. By Definition 2.4, $f^{-1}(V)$ is α-open in X and contains x. Hence $f : (X, \tau^\alpha) \to (Y, \upsilon)$ is strongly M-precontinuous.

(b)$\Rightarrow$(c). Let $x \in X$ and V be any preopen set of Y containing $f(x)$. Set $U = f^{-1}(V)$, then by (b), U is an α-open subset of X containing x and $f(U) \subset V$.

(c)$\Rightarrow$(d). Let V be any preopen set of Y and $x \in f^{-1}(V)$. By (c), there exists an α-open set U of X containing x such that $f(U) \subset V$. Thus, we have $x \in U \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(U))) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))))$ and hence $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))))$.

(d)$\Rightarrow$(e). Let F be any preclosed subset of Y. Set $V = Y - F$, then V is preopen in Y. By (d), we obtain $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))))$ and hence $f^{-1}(F) = X - f^{-1}(Y - F) = X - f^{-1}(V)$ is α-closed in X.

(e)$\Rightarrow$(f). Let B be any subset of Y. Since $\mathrm{pCl}(B)$ is a preclosed subset of Y, then $f^{-1}(\mathrm{pCl}(B))$ is α-closed in X and hence $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{pCl}(B)))))$ $\subset f^{-1}(\mathrm{pCl}(B))$. Therefore, we obtain $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(B)))) \subset f^{-1}(\mathrm{pCl}(B))$.

(f)$\Rightarrow$(g). Let A be any subset of X. By (f), we have $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(f(A))))) \subset f^{-1}(\mathrm{pCl}(f(A)))$ and hence $f(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))) \subset \mathrm{pCl}(f(A))$.

(g)$\Rightarrow$(a). Let V be any preopen subset of Y. Since $f^{-1}(Y - V) = X - f^{-1}(V)$ is a subset of X and by (g), we obtain $f(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(Y - V))))) \subset \mathrm{pCl}(f(f^{-1}(Y - V))) \subset \mathrm{pCl}(Y - V) = Y - p\mathrm{Int}(V) = Y - V$ and hence $X - \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(V)))) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(X - f^{-1}(V)))) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(Y - V)))) \subset f^{-1}(f(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(Y - V)))))) \subset f^{-1}(Y - V) = X - f^{-1}(V)$. Therefore, we have $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(V))))$ and hence $f^{-1}(V)$ is α-open in X. Thus, f is α-preirresolute.

LEMMA 3.1 (Chae et al.[6], El-Deeb et al. [7] and Abd El-Monsef et al. [2]). *Let $\{X_\lambda : \lambda \in \Lambda\}$ be a family of spaces and U_{λ_i} be a nonempty subset of*

X_{λ_i} for each $i = 1, 2, \ldots, n$. Then $U = \prod_{\lambda \neq \lambda_i} X_\lambda \times \prod_{i=1}^{n} U_{\lambda_i}$ is a nonempty α-open [6] (resp. preopen [7], β-open [2]) subset of $\prod X_\lambda$ if and only if U_{λ_i} is α-open (resp. preopen, β-open) in X_{λ_i} for each $i = 1, 2, \ldots, n$.

THEOREM 3.2. *A function $f : X \to Y$ is α-preirresolute if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is α-preirresolute.*

Proof. Let $x \in X$ and V be any preopen set of Y containing $f(x)$. Then $X \times V$ is a preopen set of $X \times Y$ by Lemma 3.1 and contains $g(x)$. Since g is α-preirresolute, there exists an α-open set U of X containing x such that $g(U) \subset X \times V$ and hence $f(U) \subset V$. Thus f is α-preirresolute.

THEOREM 3.3. *A function $f : X \to \prod Y_\lambda$ is α-preirresolute, then $P_\lambda \circ f : X \to Y_\lambda$ is α-preirresolute for each $\lambda \in \Lambda$, where P_λ is the projection of $\prod Y_\lambda$ onto Y_λ.*

Proof. Let V_λ be any preopen set of Y_λ. Since P_λ is continous open, it is preirresolute [13, Theorem 3.4]. Therefore, $P_\lambda^{-1}(V_\lambda)$ is preopen in $\prod Y_\lambda$. Since f is α-preirresolute, then $f^{-1}(P_\lambda^{-1}(V_\lambda)) = (P_\lambda \circ f)^{-1}(V_\lambda)$ is α-open in X. Hence $P_\lambda \circ f$ is α-preirresolute for each $\lambda \in \Lambda$.

THEOREM 3.4. *If the product function $f : \prod X_\lambda \to \prod Y_\lambda$ is α-preirresolute, then $f_\lambda : X_\lambda \to Y_\lambda$ is α-preirresolute for each $\lambda \in \Lambda$.*

Proof. Let $\lambda_0 \in \Lambda$ be an arbitrarily fixed index and V_{λ_0} be any preopen set of V_{λ_0}. Then, $\prod Y_\gamma \times V_{\lambda_0}$ is preopen in $\prod Y_\gamma$ by Lemma 3.1, where $\lambda_0 \neq \gamma \in \Lambda$. Since f is α-preirresolute, then $f^{-1}(\prod Y_\gamma \times V_{\lambda_0}) = \prod X_\gamma \times f_{\lambda_0}^{-1}(V_{\lambda_0})$ is α-open in $\prod X_\lambda$ and hence, by Lemma 3.1, $f_{\lambda_0}^{-1}(V_{\lambda_0})$ is α-open in X_{λ_0}. This implies that f_{λ_0} is α-preirresolute.

THEOREM 3.5. *If $f : (X, \tau) \to (Y, \upsilon)$ is α-preirresolute and A is a preopen subset of X, then the restriction $f/A : A \to Y$ is α-preirresolute.*

Proof. Let V be any preopen set of Y. Since f is α-preirresolute, then $f^{-1}(V)$ is α-open in X. Since A is preopen in X, $(f/A)^{-1}(V) = A \cap f^{-1}(V)$ is α-open in A [13, Lemma 1.1]. Hence f/A is α-preirresolute.

THEOREM 3.6. *Let $f : (X, \tau) \to (Y, \upsilon)$ be a function and $\{A_\lambda : \lambda \in \Lambda\}$ be a cover of X by α-open sets of (X, τ). Then f is α-preirresolute if $f/A_\lambda : A_\lambda \to Y$ is α-preirresolute for each $\lambda \in \Lambda$.*

Proof. Let V be any preopen set of Y. Since f/A_λ is α-preirresolute, $(f/A_\lambda)^{-1}(V)$ is α-open in A_λ. Since A_λ is α-open in X, $(f/A_\lambda)^{-1}(V)$ is α-open in X for each $\lambda \in \Lambda$ [13, Lemma 1.2]. Therefore, $f^{-1}(V) = X \cap f^{-1}(V) = \cup\{A_\lambda \cap f^{-1}(V) : \lambda \in \Lambda\} = \cup\{(f/A_\lambda)^{-1}(V) : \lambda \in \Lambda\}$ is

α-open in X because the union of α-open sets is an α-open set. Hence f is α-preirresolute.

THEOREM 3.7. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition $g \circ f : X \to Z$ is α-preirresolute if f is α-preirresolute and g is preirresolute.*

Proof. Let W be any preopen subset of Z. Since g is preirresolute, $g^{-1}(W)$ is preopen in Y. Since f is α-preirresolute, then $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is α-open in X and hence $g \circ f$ is α-preirresolute.

4. β-preirresolute functions

Now, the proofs of the following four theorems are similar to those of theorems in section 3 and are thus omitted.

THEOREM 4.1. *For a function $f : (X, \tau) \to (Y, \upsilon)$, the following are equivalent:*

(a) *f is β-preirresolute;*

(b) *For each $x \in X$ and each preopen set V of Y containing $f(x)$, there exists a β-open set U of X containing x such that $f(U) \subset V$;*

(c) $f^{-1}(V) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(V))))$ *for every preopen set V of Y;*

(d) *$f^{-1}(F)$ is β-closed in X for every preclosed set F of Y;*

(e) $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(f^{-1}(B)))) \subset f^{-1}(\mathrm{pCl}(B))$ *for every subset B of Y;*

(f) $f(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))) \subset \mathrm{pCl}(f(A))$ *for every subset A of X.*

THEOREM 4.2. *A function $f : X \to Y$ is β-preirresolute if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is β-preirresolute.*

THEOREM 4.3. *A function $f : X \to \prod X_\lambda$ is β-preirresolute, then $P_\lambda \circ f : X \to X_\lambda$ is β-preirresolute for each $\lambda \in \Lambda$, where P_λ is the projection of $\prod X_\lambda$ onto X_λ.*

THEOREM 4.4. *If the product function $f : \prod X_\lambda \to \prod Y_\lambda$ is β-preirresolute, then $f_\lambda : X_\lambda \to Y_\lambda$ is β-preirresolute for each $\lambda \in \Lambda$.*

THEOREM 4.5. *If $f : (X, \tau) \to (Y, \upsilon)$ is β-preirresolute and A is an α-open subset of X, then the restriction $f/A : A \to Y$ is β-preirresolute.*

Proof. Let V be any preopen set of Y. Since f is β-preirresolute, then $f^{-1}(V)$ is β-open in X. Since A is α-open in X, $(f/A)^{-1}(V) = A \cap f^{-1}(V)$ is β-open in A [1, Lemma 2.5]. Hence f/A is β-preirresolute.

THEOREM 4.6. *Let $f : (X, \tau) \to (Y, \upsilon)$ be a function and $\{A_\lambda : \lambda \in \Lambda\}$ be a cover of X by β-open sets of (X, τ). Then f is β-preirresolute if $f/A_\lambda : A_\lambda \to Y$ is β-preirresolute for each $\lambda \in \Lambda$.*

Proof. Let V be any preopen set of Y. Since f/A_λ is β-preirresolute, $(f/A_\lambda)^{-1}(V)$ is β-open in A_λ. Since A_λ is β-open in X, $(f/A_\lambda)^{-1}(V)$ is β-open in X for each $\lambda \in \Lambda$ [1, Lemma 2.7]. Therefore, $f^{-1}(V) = X \cap f^{-1}(V) = \cup\{A_\lambda \cap f^{-1}(V) : \lambda \in \Lambda\} = \cup\{(f/A_\lambda)^{-1}(V) : \lambda \in \Lambda\}$ is β-open in X because the union of β-open sets is a β-open set. Hence f is β-preirresolute.

THEOREM 4.7. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition $g \circ f : X \to Z$ is β-preirresolute if f is β-preirresolute and g is preirresolute.*

Proof. The proof is similar to that of Theorem 3.7 and is thus omitted.

We recall that a space X is said to be *submaximal* if every dense subset of X is open in X and X is *extremally disconnected* if the closure of each open subset of X is open in X.

THEOREM 4.8. *Let (X, τ) be a submaximal and extremally disconnected space. Then, for a function $f : (X, \tau) \to (Y, \upsilon)$, we have*

(a) *Strongly M-precontinuity $\Leftrightarrow$ α-preirresoluteness $\Leftrightarrow$ preirresoluteness $\Leftrightarrow$ β-preirresoluteness.*

(b) *Strongly α-irresoluteness $\Leftrightarrow$ α-irresoluteness $\Leftrightarrow$ α-precontinuity $\Leftrightarrow$ almost α-irresoluteness.*

Proof. This follows from the fact that if (X, τ) is a submaximal and extremally disconnected space, then $\tau = \tau^\alpha = \mathrm{PO}(X) = \beta\mathrm{O}(X)$ (Janković [8] and Nasef and Noiri [14]).

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