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Boundary rigidity for Riemannian manifolds

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Summary. In the work question of the uniqueness of the solution of the problem of restoring the Riemannian metric by the distances between the pairs of the points of boundary of the region are investigated. The uniqueness of solution of the problem, up to the diffeomorphism identical on the boundary of the region, for the sufficiently wide class of the metrics is proven.

Key words: Hodograph, semigeodesic coordinates, inverse kinematic problem, integral geometry problem, special kinetic equation

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In this paper the problem of reconstructing a Riemannian metric $g = (g_{ij}(x))$ in the bounded domain D of space R^n, $(n > 1)$ with the boundary S of class C^3 from the distances between the boundary points of the domain in a metric, and connected with it problems are investigated. Here questions connected with the uniqueness of a solution to the problem in question will be discussed. A domain D is called convex with respect to a metric g, if any two points $x_0, x \in \bar{D}$, can be joint by unique geodesic of this metric all points of which with the exception, of possible ends, belong to domain D.

For the points $x_0, x \in S$ let us denote through $H_g(x, x_0)$ the distance between the points x_0, x in the metric g. Function $H_g(x, x_0)$ determined on the set $S \times S$ is called the hodograph of the metric g.

Problem 1. Determine a metric g in the region D if the hodograph $H_g(x, x_0)$ is known.

Problem 1 arose in geophysics in connection with the study of distribution of the velocities of propagation of elastic waves inside terrestrial globe and its linearization has, in particular, applications in tomography.

It is easy to show nonuniqueness of a solution of the problem 1. Indeed, let φ be the diffeomorphism of region D to itself from class C^1 identical on S. It transforms each simple metric g_1 again into a simple metric $g_2 = \varphi^* g_1$, in the sense that for any vectors $\xi, \eta \in T_x D$, $\langle \xi, \eta \rangle_x^{(2)} = \langle \varphi_* \xi, \varphi_* \eta \rangle_{\varphi(x)}^1$ equality holds, where φ_*- the differential of the map φ, $\langle ., . \rangle_x^i$- scalar product on $T_x D$ is determined by the metric $g_i, i = 1, 2$, D is convex with respect to a metric g_i. These two metrics have different families of geodesic, but the same hodograph.

The questions naturally arise:

1) are there other types of the nonuniqueness of solution of problem 1 ?

2) when is a metric determined by its hodograph up to isometry, identical on ?

3) for what classes of metrics hodograph determines a metric uniquely?

Let us clarify the formulation of problem 1 as follows:

Problem 2. Let g_1, g_2 be two metrics which are convex in D. Does the existence of a diffeomorphism $\varphi : D \to D$ follow from the equality $H_1(x, x_0) = H_2(x, x_0)$, such that $\varphi|_S = 1$, and $g_2 = \varphi^* g_1$, where $H_k(x, x_0)$ the hodograph of metric $g_k, k = 1, 2$ and equality $\varphi|_S = 1$ it means that the mapping φ is identical on S?

Positive answer to the question formulated in problem 2, is obtained only for few class of metrics (see [1-11]). Below (in theorems 1, 2, 3) it is assumed that the domain D is convex with respect to a metric $g_k = (g_{ij}^{(k)}(x)) \in C^4(\bar{D}), k = 1, 2$. Furthermore it is assumed that a metrics g_k are known on the set D_ε / D and coincide on D_ε / D, where D_ε is the neighborhood of set $D, \varepsilon > 0$, i.e $D_\varepsilon = \{x \in R^n / d(x, D) < \varepsilon\}$, $d(x, D)$- the Euclidean distance between the point $x \in R^n$ and the set D, $d(x, D) = \inf |x - y|$, $y \in D$. Let us note that the last condition is not, generally speaking, restriction on metrics g_k in D if they hodographs coincide. Indeed, it is proved in [10] that, if H_1 and H_2 coincide, then in the suitable coordinates g_1 and g_2 will coincide in the space $C^2(S)$. Consequently, it is possible to continue g_2 from the boundary $S \in C^3$ to D_ε / D by equality $g_2 = g_1$. Then the metrics g_1 and $\bar{g}_2$ will be from $C^2(D_\varepsilon)$, have the same hodograph and coincide on D_ε / D, where $\bar{g}_2 = g_2$ when $x \in D$, and $\bar{g}_2 = g_1$ when $x \in D_\varepsilon / D$.

Theorem 1. *Let $H_1 = H_2$, then*

a) there exists a diffeomorphism $\varphi : D \to D$, that $\varphi|_S = 1$ and $g_2 = \varphi^* g_1$, $\varphi \in C^3(D)$,

b) if $g^k_{1i} = \delta_{1i}$, where $i = 1, 2, \dots, n$; $k = 1, 2$; δ_{1i} - Kronecker's symbols, then metrics g_1 and g_2 coincide in D.

Let $V(x, t)$ be a solution of the equation

$$V_{tt} - \sum_{i,j=1}^{n} a_{ij}(x) V_{x_i x_j} = 0 \tag{1}$$

satisfying the conditions

$$V, V_t|_{t=0} = 0, \frac{\partial V}{\partial N}|_{\partial D x(0,T)} = f(\gamma, t), \tag{2}$$

where f continuous on $Sx(0, T)$, $N = N(\gamma)$ is outer normal to $Sx(0, T)$, $\gamma \in S$, $t \in (0, T)$, $T \geq T_0$, T_0 -the diameter of the region D in the metric g, $(a_{ij}(x))$ inverse to $(g_{ij}(x))$ matrix.

Let us assume that on $C(Sx(0, T))$ the operator of reaction is known: $R_a f = V(\gamma, t)$, where $a = (a_{ij}(x))$.

Problem 3. It is necessary to determine vector function $(a_{ij}(x))_1^n$ from equation (1) in the region D, if the operator R_a is known.

Theorem 2. *Let*

1) the domain D is convex with respect to a metric $(g^k_{ij}(x))$, where $(g^k_{ij}(x))$ is the inverse of $(a^{(k)}_{ij}(x))$ matrix

2) the operators of reaction corresponding to matrix- functions $a_k = (a^{(k)}_{ij}(x))$, $k = 1, 2$ coincide, i.e $R_{a_1} = R_{a_2}$. Then

a) there exists a diffeomorphism $\varphi : D \to D$ from class $C^3(D)$, such that $\varphi|_S = 1$ and $a_2 = \varphi^* a_1$.

b) if $a^{(k)}_{11} = 1$, $a^{(k)}_{1i} = 0$, $k = 1, 2; i = 2, 3, \dots, n$ then $a^{(1)}_{ij}(x) = a^{(2)}_{ij}(x)$ in the region $D, i, j = 2, 3, \dots, n$.

Problem 1 is connected the following so-called inverse kinematic

Problem 4. It is necessary to find a vector function $a = (a_{ij}(x))_1^n$ in D if, $\tau(x, x_0)$ is known for points $x, x_0 \in S$ and satisfies in D the equation

$$\sum_{i,j=1}^{n} a_{ij}(x) \tau_{x_i} \tau_{x_j} = 1, \tag{3}$$

and the condition $\tau(x, x_0) = O(|x - x_0|)$.

It is not difficult to see that (3) is the characteristic equation of equation (1), when the corresponding characteristic surface is represented in the form $t = \tau(x, x_0)$, where $\tau(x, x_0)$(see [1]) is the distance between the points x and x_0 in the metric g.

Theorem 3. *Let the condition 1) of the theorem 2 be satisfied. Then if $H_1 = H_2$ then*

a) there exists a diffeomorphism $\varphi : D \to D$ from class $C^3(D)$, such that $\varphi|_S = 1$ and $a_2 = \varphi^* a_1$.

b) under the conditions $a^{(k)}_{11} = 1$, $a^{(k)}_{1i} = 0$, $k = 1, 2; i = 2, 3, \dots, n$ matrix - functions a_1 and a_2 coincide in D.

Assertion b) of theorem 3 is the key result of this paper and there is an improvement of the theorem 1 of work [5], namely, in this paper the assertion b) is proved without restriction to a metric g of the form $(x \in \bar{D},\ \xi \in R^n)$

$$-\frac{1}{2}\frac{\partial}{\partial x_1}\sum_{i,j=2}^{n} g_{ij}\xi^i\xi^j \geq \alpha_0|'\xi|^2, \alpha_0 > 0,' \xi = (\xi^2, \xi^3, \dots, \xi^n).$$

Let us give the outline of the proof of assertion b) of theorem 3. For this let us introduce the following notations relating to the metric $g_k = (g^{(k)}_{ij}(x))$: $\Gamma_k(x, x_0)$ - ray connecting the points $x_0 \in \partial D$, $x \in \bar{D}$, a $\tau_k(x, x_0)$ - the distance between these points; $\gamma_2(x, \zeta)$ - the ray of metric g_2 starting from the point $x \in D$ in the direction ζ , where $(i = 1, 2, \dots, n)$

$$\zeta = (\zeta^1, \zeta^2, \dots, \zeta^n), \zeta^i = \sum_{j=1}^{n} a^{(2)}_{ij}(x)p^j_0, p^i_0 = (\tau_1(x, x_0) + \tau_2(x, x_0))_{x_i}.$$

Let us assume that under the conditions $a^{(k)}_{11} = 1$, $a^{(k)}_{1i} = 0$, $i = 2, 3, \dots, n$ problem 4 has two solutions $a_k = (a^{(k)}_{ij})$, $k = 1, 2$, with the same data, i.e. $H_1 = H_2$ Then in equation (3) if we first take $a_{ij} = a^{(2)}_{ij}$, and than $a_{ij} = a^{(1)}_{ij}$, and subtracting the resulting equations from each other and transforming the obtained equation correspondingly, for the functions $d(x, x_0) = \tau_2(x, x_0) - \tau_1(x, x_0)$ and $b_{ij} = a^{(2)}_{ij}(x) - a^{(1)}_{ij}(x)$ we have

$$\sum_{i,j=1}^{n} a^{(2)}_{ij} p^i_0 d_{x_j} + \sum_{i,j=2}^{n} b_{ij}\tau_{1x_i}\tau_{1x_j} = 0. \tag{4}$$

Let us note that, since $a^{(k)}_{1j} = 0$, $j = 2, 3, \dots, n$, $a^{(k)}_{11} = 1$, we have $b_{1j} = 0$, $j = 1, 2, \dots, n$. It is easy to see that the expression

$\sum_{i,j=1}^{n} a_{ij}^{(2)} p_0^i d_{x_j}$ is a derivative of $d(x,x_0)$ along $\gamma_2(x,\zeta)$. Integrating equality (4) along the ray $\gamma_2(x,\zeta)$ and taking into account that, for the points $x, x_0 \in S$, $d(x,x_0) = 0$, we will obtain

$$0 \equiv d(x,x_0) = \int_{\gamma_2(x,\zeta)} \sum_{i,j=2}^{n} b_{ij} \tau_{1z_i}(z,x_0)\tau_{1z_j}(z,x_0)dt. \tag{5}$$

Recalling, that (see [1])

$$v_i^{(1)}(z,x_0) \equiv \frac{dz_{(1)}^{(i)}}{dt} = \sum_{j=1}^{n} a_{ij}^{(1)}(z)\tau_{1z_j}(z,x_0), i = 1,2,\dots,n,$$

from (5) we have

$$d(x,x_0) = \int_{\gamma_2(x,\zeta)} \sum_{k,l=2}^{n} c_{kl} v_k^{(1)}(z,x_0) v_l^{(1)}(z,x_0)dt, \tag{6}$$

where $c_{kl} = \sum_{i,j=2}^{n} b_{ij} g_{ik}^{(1)} g_{jl}^{(1)}, x, x_0 \in S$.

Equation (6) is a problem of integral geometry for the matrix-function (c_{kl}) but instead of it we will investigate the inverse problem for the special kinetic equation connected with it. In order to formulate it, we need some notations. Let rays $\Gamma_2(x,x_0)$, $\Gamma_1(x,x_0)$ leave from the point $x \in D$ at angles ξ, $f(\xi)$ correspondingly. In view of the condition on a metric $g_k(x), k = 1,2$ for each fixed $x \in D$ functions $\xi = \xi(x,x_0)$, $\eta = f(\xi) \in C^3(S_1^{(2)})$, are invertible and $x_0 = x_0(\xi) \in C^3(S_1^{(2)})$, $\xi = f^{-1}(\eta)$ where $S_1^{(2)}$, is the sphere of radius 1 of the metric $g_2(x)$ at point x.

Furthermore Jacobian $|\frac{\partial f(\xi)}{\partial \xi}| > 0$ and $x_0 \in S$. Using these facts it is possible to prove that for each fixed $x \in D$ the equation

$$F(\xi) \equiv \xi + f(\xi) = \zeta \tag{7}$$

is invertible $\xi = F^{-1}(\zeta)$, where $\xi \in S_1^{(2)}$, $\zeta = F(\xi) \in S_v^{(2)}$, $S_v^{(2)}$ - the sphere of a radius $v(\xi) = (\sum_{i,j=1}^{n} a_{ij}^{(2)}(x)p_0^i p_0^j)^{1/2}$ of the metric $g_2(x)$, $F^{-1}(\zeta)$ in the domain of definition. Moreover using the condition that metric $g_k(x)$ is written in semigeodesic coordinates, the existence of a function $'I('\zeta) = (I^2('\zeta),\dots,I^n('\zeta)) \in C^3$ such, that

$$v_i^{(1)}(x,x_0) = I^i('\zeta), i = 2,3,\dots,n \tag{8}$$

with

$$'\zeta \to 0,\ 'I('\zeta) \to 0, \tag{9}$$

where $R^n_0 = R^n/\{0\}$ is R^n without the origin of coordinates, can be proven. Here and subsequently through $'\xi('\zeta)$ is denoted the vector $\xi \in R^n$ $(\zeta \in R^n)$ without the first component. In this case if function $f(\xi)$ is continued from the set $S_1^{(2)}$ by the formula $f(l\xi) = lf(\xi)$, $(l > 0,\ \xi \in S_1^{(2)}$) then functions $F(\xi)$, $F^{-1}(\zeta)$, $'I('\zeta)$ will be also homogenous functions.

Let us introduce a function $u(x, \zeta)$ according to the formula

$$u(x,\zeta) = \int_{\gamma_2(x,\zeta)} \sum_{i,j=2}^{n} c_{ij} I^i('\dot{z}) I^j('\dot{z}) dt,$$

where $'\dot{z} = (\frac{dz^2}{dt}, \frac{dz^3}{dt}, \ldots, \frac{dz^n}{dt})$. Differentiating $u(x, \zeta)$ at point x in the direction ζ we have the following kinetic equation

$$\sum_{i=1}^{n} \zeta^i u_{x_i} + \sum_{i,j,k=1}^{n} \Gamma^k_{ij} \zeta^i \zeta^j u_{\zeta^k} = \sum_{i,j=2}^{n} c_{ij} I^i('\zeta) I^j('\zeta), \tag{10}$$

where Γ^k_{ij} - Christoffel symbols of the metric g_2. It follows from equality (6) and (8) that for any $x \in S$ we have $u(x, \zeta) = 0$.

Thus for the determination $u(x, \zeta)$ and a matrix- function (c_{ij}), we have the inverse problem (10), $u(x, \zeta) = 0$. Modifying work technique [5] from relations (10), $u(x, \zeta) = 0$ using the condition (9), it is proven, that $c_{ij} = 0$, i.e. $a^{(1)}_{ij} = a^{(2)}_{ij}$, $i, j = 1, 2, \ldots, n$. Assertion b) of theorem 3 is proved.

Outline of the proof of theorem 1. Assertion b) of theorem 1 follows from assertion b) of theorem 3. Let us prove now assertion a) of theorem 1. Let problem 2 have two solutions, with the same data i.e.,$H_1 = H_2$. In the region D for the metric g_k, $k = 1, 2$, we introduce semigeodesic coordinates as follows. Let us select any point $V_0 \in D_\varepsilon/\bar{D}$ and let us consider the geodesics outgoing from it. The so-called geodesic hyperspheres orthogonally intersect these geodesics. We take the ends of the segments of a constant length $s_k = r$ on the geodesics, outgoing from V_0. These ends form the hypersurface, which is called the geodesic hypersphere of a radius r with center in V_0 of the metric g_k. Let us examine a certain region on the hypersphere (which lies outside $\bar{D}$) with the parameters $u^1_k, u^2_k, \ldots, u^{n-1}_k$. We will carry the geodesics to the same parameters, connecting center hyperspheres V_0 with the points of region D. We will characterize position of an arbitrary point L on the geodesic with arc length $s_k = V_0L$. Then it is obvious that in view of the condition on the metric g_k the variables $u^1_k, u^2_k, \ldots, u^{n-1}_k, s_k$ form the semigeodesic coordinate system in the region D (see [12]). We will denote subsequently coordinates

$u_k^1, u_k^2, \ldots, u_k^{n-1}, s_k$ through $x_1^k, x_2^k, \ldots, x_n^k$, or $x = (x_1^k, x_2^k, \ldots, x_n^k)$. Let D_k be the domain of semigeodesic coordinates of the metric g_k we introduced in the region D, $k = 1, 2$. It is easy to establish one-to-one correspondence between the regions D_1 and D_2 as follows: To each point $x^{(1)} \in D_1$ one assigns a point $x^{(2)} \in D_2$ so that both points become images of the same point in the region D in the semigeodesic coordinates with respect to corresponding metrics g_1 and g_2. Then it is unnecessary to construct the independent coordinate system in each domain D_1, D_2. Namely, it is possible to transform a coordinate system in the region D_2 into the region D_1 as follows: To each point $x^{(1)}$ in the region D_1, the same coordinates x_i are assigned which are already a coordinates in the region D_2 of the corresponding point $x^{(2)}$.Thus, for the corresponding points $x^{(1)}$ and $x^{(2)}$ we have $x_i^{(1)} = x_i^{(2)}$, $i = 1, 2, \ldots, n$. In this case boundary of the region D_1 is mapped into boundary of the region D_2. Nevertheless, generally speaking, metrics in both spaces remain different. This means that, from analytical point of view there is one region $\tilde{D}$ in which we have two different metrics $\tilde{g}_1$, $\tilde{g}_2$. Moreover both metrics are written down in the semigeodesic coordinates in $\tilde{D}$. It is known that (see [12]) $\tilde{g}_k$ is written down in the semigeodesic coordinates in $\tilde{D}$ if and only if $\tilde{g}_{1i}^k = \delta_{1i}$, where $i = 1, 2, \ldots, n$; $k = 1, 2$; δ_{1i} - Kronecker's symbols. By the condition of theorem 1 on the metrics $\tilde{g}_1, \tilde{g}_2$ in the region $\tilde{D}$. they have the same hodograph, therefore, by assertion b) of theorem 3 they coincide, i.e. $\tilde{g}_1 = \tilde{g}_2$.

Let us build diffeomorphism φ according to equalities $x_i^{(1)} = x_i^{(2)}$, $i = 1, 2, \ldots, n$, as follows: let, us assign to a point $x^1 \in D$ with the semigeodesic coordinates $(u_1^1, u_1^2, \ldots, u_1^{n-1}, s_1)$ in the metric g_1 the point $x^2 \in D$ with the semigeodesic coordinates $(u_2^1, u_2^2, \ldots, u_2^{n-1}, s_2)$ in the metric g_2 if the equalities $u_2^i = u_1^i$, $1, 2, \ldots, n$, $s_2 = s_1$ hold. It is not difficult to see that $\varphi|_S = 1$. Actually, for the point $x \in S$ the rays $\varGamma_1(x, V_0)$, $\varGamma_2(x, V_0)$ outside of the region D coincide, since outside D we have $g_1 = g_2$, where $\varGamma_k(x, V_0)$ - geodesic of the metric g_k connecting points $x \in D$ and V_0, $k = 1, 2$. Here, in construction of semigeodesic coordinates for the metrics g_1 and g_2 we take the same geodesic hypersphere with the center at the point V_0 which lies outside D. Then by the uniqueness of the ray $\varGamma_k(x, V_0)$ and from the definition of the coordinates $u_1^1, u_1^2, \ldots, u_1^{n-1}$, the first $(n-1)$ components of the semigeodesic coordinates of the point $x \in S$ in the metrics g_1 and g_2 coincide, i.e $u_2^i = u_1^i$, $1, 2, \ldots, n-1$. But the equality of last components ($s_2 = s_1$) follows from the equality $H_1 = H_2$ and from the fact that rays $\varGamma_1(x, V_0)$ and $\varGamma_2(x, V_0)$ outside of D coincide. Consequently, we have:

1) for $x \in S$, $\varphi(x) = x$,

2) the regions D_1 and D_2 are coincide, i.e. $D_1 = D_2 = \tilde{D}$.

Then taking into account 2), the convexity of domain D with respect to g_k, $k = 1, 2$, and determination of φ we will obtain, that φ transforms D to itself. According to the theorem about continuous-differentiability dependence of the solution to the Cauchy problem (determining the ray of the metric $g_{(k)} \in C^{(4)}(D)$, $k = 1, 2$) from the initial data and to condition $S \in C^3$ and also from the determination of the φ we have, that $\varphi \in C^3(D)$. Equality $\tilde{g}_1 = \tilde{g}_2$, the determination of mappings φ and φ^* give us, $g_2 = \varphi^* g_1$. This proves Theorem 1.

Assertion a) of Theorem 3 follows from the proof of assertion a) of Theorem 1. Theorem 2 is proven using the properties of the fundamental solution of equation (1) and the assertion of theorem 1.

For the number of the inverse problems of the theory of scattering and spectral analysis of those connected with problem 2, from theorem 1 it is possible to obtain the corresponding results.

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