Computers and Mathematics with Applications 59 (2010) 898–902

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On the asymptotic behavior and periodic nature of a difference equation with maximum

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ARTICLE INFO

Article history:
Received 14 March 2009
Received in revised form 25 August 2009
Accepted 7 October 2009

Keywords:
Max operator
Difference equation
Positive solution
Global attractivity
Periodicity

ABSTRACT

In this paper, we investigate the asymptotic behavior and periodic nature of positive solutions of the difference equation

$$x_n = \max\left\{\frac{A}{x_{n-1}}, \frac{1}{x_{n-2}^{\alpha}}\right\}, \quad n \geq 0,$$

where $A \geq 0$ and $0 \leq \alpha \leq 1$. We prove that every positive solution of this difference equation approaches $\bar{x} = 1$ or is eventually periodic with a period 2, 3 or 4.

1. Introduction

The max operator arises naturally in certain models in automatic control theory (see [1,2]). In recent years, the discrete case involving difference equations with maximum has been receiving increasing attention; see e.g., [3–20,1,21,2,22–29] (see also the references therein).

It was proved that every positive solution of the difference equation

$$x_n = \max\left\{\frac{A}{x_{n-p}^{\alpha}}, \frac{B}{x_{n-k}^{\beta}}\right\}, \quad n \geq 0, \tag{1.1}$$

where p, k are positive integers, $0 < \alpha < 1, 0 < \beta < 1, 0 < A$ and $0 < B$, approaches $\bar{x} = \max\left\{A^{\frac{1}{\alpha+1}}, B^{\frac{1}{\beta+1}}\right\}$ in [15].

In [25], it was proved that every positive solution of the difference equation

$$x_n = \max\left\{\frac{A}{x_{n-1}^{\alpha}}, \frac{B}{x_{n-2}^{\beta}}\right\}, \quad n \geq 0, \tag{1.2}$$

where $0 < \alpha < 1, 0 < \beta < 1, 0 < A$ and $0 < B$, approaches $\bar{x} = \max\left\{A^{\frac{1}{\alpha+1}}, B^{\frac{1}{\beta+1}}\right\}$.

The asymptotic behavior of positive solutions of the difference equation

$$x_n = \max\left\{\frac{1}{x_{n-1}^{\alpha}}, \frac{A}{x_{n-2}}\right\}, \quad n \geq 0, \tag{1.3}$$

where $0 < \alpha < 1$ and $0 < A$, was investigated in [29]. It was shown that every positive solution of this difference equation approaches $\bar{x} = 1$ or is eventually periodic with a period 4. Also, the author of [29] proposed the following conjecture.

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doi:10.1016/j.camwa.2009.10.004

Conjecture 1. *Consider the difference equation*

$$x_n = \max\left\{\frac{A_1}{x_{n-1}^{\alpha_1}}, \frac{A_2}{x_{n-2}^{\alpha_2}}, \dots, \frac{A_p}{x_{n-p}^{\alpha_p}}\right\}, \quad n \geq 0, \tag{1.4}$$

where $A_1 \geq 0, A_2 \geq 0, \dots, A_p \geq 0$, $0 \leq \alpha_1 \leq 1$, $0 \leq \alpha_2 \leq 1, \dots, 0 \leq \alpha_p \leq 1$, $\max\{\alpha_1, \alpha_2, \dots, \alpha_p\} = 1$. Let $\alpha_q = 1$. If $A_q > \max\{A_j : 1 \leq j \leq p, j \neq q\}$, then every positive solution of (1.2) is eventually periodic with a period $T = 2q$.

It is interesting that the behavior of their solutions is very different although the difference equation (1.2) and (1.3) are very similar. Motivated by this result, we investigate the asymptotic behavior and periodic nature of positive solutions of the difference equation

$$x_n = \max\left\{\frac{A}{x_{n-1}}, \frac{1}{x_{n-2}^{\alpha}}\right\}, \quad n \geq 0, \tag{1.5}$$

where $0 \leq A$, $0 \leq \alpha \leq 1$ and the initial conditions x_{-1}, x_{-2} are positive real numbers. We prove every positive solution of this difference equation approaches $\bar{x} = 1$ or is eventually periodic with a period 2, 3 or 4. We believe that the method used in this paper may be used for confirming Conjecture 1.

2. Preliminaries

Amleh et al. proved that every positive solution of the difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{B}{x_{n-1}}\right\} \tag{2.1}$$

is eventually periodic with a period T, where $T = 2$ provided $A > B \geq 0$, $T = 4$ provided $B > A \geq 0$ and $T = 3$ provided $A = B > 0$. So, we conclude that every positive solution of Eq. (1.5) with $\alpha = 1$ and $A \geq 0$ is eventually periodic with a period 2, 3 or 4. The proof of this result was given in [3] and will be omitted.

Assume that $\alpha = 0$ and $A \geq 0$. Eq. (1.5) with $\alpha = 0$ and $A \geq 0$ has the form

$$x_n = \max\left\{\frac{A}{x_{n-1}}, 1\right\}, \quad n \geq 0. \tag{2.2}$$

By iteration, from Eq. (2.2) we get that every positive solution of Eq. (1.5) with $\alpha = 0$ is eventually periodic with a period 2 in the case $A > 1$. In the case $0 \leq A < 1$, Eq. (1.5) with $\alpha = 0$ becomes $x_n = 1$, for $n \geq 1$.

Assume that $A = 0$ and $0 < \alpha < 1$. Then, from Eq. (1.5) we obtain immediately

$$x_n = \frac{1}{x_{n-2}^{\alpha}} \quad \text{for } n \geq 0. \tag{2.3}$$

It has been proved that every solution to the difference equation $x_n = \frac{A}{x_{n-m}^{\alpha}}$, $0 < \alpha < 1, A > 0$ converges to $\bar{x} = A^{\frac{1}{\alpha+1}}$ in [25]. So, every positive solution of the Eq. (2.3) approaches $\bar{x} = 1$ and its proof will be omitted.

3. The case $A = 1$

We consider the difference equation

$$x_n = \max\left\{\frac{1}{x_{n-1}}, \frac{1}{x_{n-2}^{\alpha}}\right\}, \quad n \geq 0, \tag{3.1}$$

where $0 < \alpha < 1$.

Theorem 1. *If x_n is a positive solution of Eq. (3.1), then x_n approaches $\bar{x} = 1$.*

Proof. Choose a number B such that $0 < B < 1$, let $x_n = B^{y_n}$ for $n \geq -2$. Then, Eq. (3.1) implies the difference equation

$$y_n = \min\{-y_{n-1}, -\alpha y_{n-2}\}, \quad n \geq 0, \tag{3.2}$$

where $0 < \alpha < 1$ and initial conditions are real numbers.

Let y_n be a solution of Eq. (3.2). Then it suffices to prove $y_n \to 0$. Observe that there exists a positive integer N such that $y_{3n+N} \geq 0$ for $n \geq 0$. By direct computation, we get that, $y_{3n+N-1} \leq 0$, $y_{3n+N-2} \leq 0$, $y_{3n+N+1} = -y_{3n+N}$, $y_{3n+N+2} = -\alpha y_{3n+N}$, $y_{3n+N+3} = \alpha y_{3n+N}$ and $y_{3n+N+1} \leq y_{3n+N+2} \leq 0 \leq y_{3n+N+3} \leq y_{3n+N}$. So, $y_{3n+N+1} \to 0$, $y_{3n+N+2} \to 0$ and $y_{3n+N} \to 0$. This implies $y_n \to 0$. □

4. The case $0 < A < 1$

We consider Eq. (1.5), where $0 < \alpha < 1$. Using the substitution $x_n = A^{y_n}$, $n \geq -2$, we get the difference equation

$$y_n = \min\{1 - y_{n-1}, -\alpha y_{n-2}\}, \quad n \geq 0, \tag{4.1}$$

where the initial conditions y_{-1}, y_{-2} are real numbers.
We need the following lemma to prove the main result.

Lemma 1. *Let y_n be a solution of Eq. (4.1). Then,*

$$|y_n| \leq \max\{|y_{n-1}| - 1, \alpha |y_{n-2}|\} \tag{4.2}$$

for all $n \geq 0$.

Proof. From Eq. (4.1), we have the following:
If $y_{n-1} \geq 0$ and $y_{n-2} \geq 0$ for some n, then $|y_n| \leq \max\{|y_{n-1}| - 1, \alpha |y_{n-2}|\}$.
If $y_{n-1} \leq 0$ and $y_{n-2} \leq 0$ for some n, then $|y_n| \leq \alpha |y_{n-2}|$.
If $y_{n-1} \geq 0$ and $y_{n-2} \leq 0$ for some n, then $|y_n| \leq \max\{|y_{n-1}| - 1, \alpha |y_{n-2}|\}$.
If $y_{n-1} \leq 0$ and $y_{n-2} \geq 0$ for some n, then $|y_n| = \alpha |y_{n-2}|$.
In general, we have

$$|y_n| \leq \max\{|y_{n-1}| - 1, \alpha |y_{n-2}|\}$$

for $n \geq 0$. □

Theorem 2. *If x_n is a positive solution of Eq. (1.5), where $0 < A < 1$ and $0 < \alpha < 1$, then x_n approaches $\bar{x} = 1$.*

Proof. Let y_n is a solution of Eq. (4.1). To prove x_n approaches one, it suffices to show that y_n approaches zero. From Lemma 1, we have

$$|y_n| \leq \max\{|y_{n-1}| - 1, \alpha |y_{n-2}|\}$$

for all $n \geq 0$. We can choose a number β such that $|y_{n-1}| - 1 \leq \beta |y_{n-1}|$ and $0 < \beta < 1$. Then from (4.2), we get that

$$|y_n| \leq \max\{\beta |y_{n-1}|, \alpha |y_{n-2}|\}, \quad n \geq 0. \tag{4.3}$$

Let $\gamma = \max\{\beta, \alpha\}$, then we get that

$$|y_n| \leq \gamma \max\{|y_{n-1}|, |y_{n-2}|\}, \tag{4.4}$$

where $0 < \gamma < 1$, for all $n \geq 0$.
From (4.4) and by iteration, we get that

$$
\begin{aligned}
|y_0| &\leq \gamma \max\{|y_{-1}|, |y_{-2}|\}, \\
|y_1| &\leq \gamma \max\{|y_{-1}|, |y_{-2}|\}, \\
|y_2| &\leq \gamma \max\{\gamma |y_{-1}|, \gamma |y_{-2}|\} \\
&\leq \gamma^2 \max\{|y_{-1}|, |y_{-2}|\}, \\
|y_3| &\leq \gamma^2 \max\{|y_{-1}|, |y_{-2}|\}
\end{aligned}
$$

and in the end the generalisation

$$|y_n| \leq \gamma^{[\frac{n+2}{2}]} \max\{|y_{-1}|, |y_{-2}|\} \tag{4.5}$$

for all $n \geq 0$. From (4.5), we get immediately that y_n approaches zero as $n \to \infty$. □

5. The case $A > 1$

Theorem 3. *Every positive solution of Eq. (1.5), where $0 < \alpha < 1$ and $A > 1$, is eventually periodic with a period 2.*

Proof. Assume that x_n is a solution of Eq. (1.5), where $A > 1$ and $0 < \alpha < 1$. Let $x_n = A^{y_n}$ for $n \geq -2$. Then, Eq. (1.5) implies the difference equation

$$y_n = \max\{1 - y_{n-1}, -\alpha y_{n-2}\}, \quad n \geq 0, \tag{5.1}$$

where the initial conditions are real numbers. Let y_n be a solution of Eq. (5.1). It suffices to show that y_n is eventually periodic with a period 2, to prove that x_n is eventually periodic with a period 2.

We shall show that there is an integer N such that $y_n \geq 0$ for all $n \geq N$. On the contrary, we assume that there is a positive integer k such that, $y_{n+k} < 0$ for all $n \geq N$. So, we have $y_{n+k-1} > 0$ and $y_{n+k-2} > 0$ from Eq. (5.1). Then, by computation we get $y_{n+k+1} > 0, y_{n+k+2} > 0$ and $y_{n+k+3} = 1 + \alpha y_{n+k}$.

If $y_{n+k+3} \geq 0$, then by computation we get $y_{n+k+4} > 0, y_{n+k+5} \geq 0$ and $y_{n+k+6} = -\alpha y_{n+k} > 0$. So, we obtain immediately $y_n \geq 0$ for $n \geq N + k + 1$.

If $y_{n+k+3} < 0$, we have $-y_{n+k} > \frac{1}{\alpha}$. Then by computation, we get that

$$y_{n+k+4} > 0, \qquad y_{n+k+5} > 0 \quad \text{and} \quad y_{n+k+6} = \left(\sum_{m=0}^{1} \alpha^{m}\right) + \alpha^{2} y_{n+k}.$$

If $y_{n+k+6} \geq 0$, we obtain immediately $y_n \geq 0$ for all $n \geq N + k + 4$ by similarity.

If $y_{n+k+6} < 0$, we get that

$$-y_{n+k} > \sum_{m=1}^{2} \frac{1}{\alpha^m}, \quad y_{n+k+7} > 0 \quad \text{and} \quad y_{n+k+8} > 0.$$

By similarity, y_{n+k+9} is either negative or not negative. But, it is a reality $\left(\sum_{m=1}^{l} \frac{1}{\alpha^m}\right) \to \infty$ as $l \to \infty$. So, there is a positive integer l $(l > k)$ such that

$$-y_{n+k} \leq \sum_{m=1}^{\frac{l-k}{3}} \frac{1}{\alpha^m} \quad \text{and} \quad y_{n+l-3} < 0.$$

Then, we obtain $y_{n+l-2} = 1 - y_{n+l-3}$, $y_{n+l-1} = -\alpha y_{n+l-3}$ and $y_{n+l} = 1 + \alpha y_{n+l-3} \geq 0$. By computation, we get immediately $y_{n+l+1} > 0, y_{n+l+2} \geq 0$ and $y_{n+l+3} = -\alpha y_{n+l-3}$. We obtain $y_n \geq 0$ for $n \geq N + l - 2$. Clearly, there is an integer N such that $y_n \geq 0$ for $n \geq N$.

Consider $y_n \geq 0$ for $n \geq N$. Then, from Eq. (5.1) we get that

$$y_{N+2} = \max\{1 - y_{N+1}, -\alpha y_N\} \geq 0.$$

If $y_{N+2} = 1 - y_{N+1}$, we get that $y_{N+3} = 1 - y_{N+2}$. So, we have $y_n = 1 - y_{n-1}$ for all $n \geq N + 2$.

If $y_{N+2} = -\alpha y_N$, then we get $y_N = 0, y_{N+1} > 1$ and $y_{N+3} = 1 - y_{N+2}$. Then, we have $y_n = 1 - y_{n-1}$ for all $n \geq N + 3$. Clearly, there is an integer N such that $y_n = 1 - y_{n-1}$ for $n \geq N$. So, y_n is eventually periodic with a period 2. This is desired. □

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