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Upper bounds for the number of spanning trees of graphs

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Abstract
In this paper, we present some upper bounds for the number of spanning trees of graphs in terms of the number of vertices, the number of edges and the vertex degrees.

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1 Introduction

Let G be a simple graph with n vertices and e edges. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. If two vertices v_i and v_j are adjacent, then we use the notation $v_i \sim v_j$. For $v_i \in V(G)$, the degree of the vertex v_i, denoted by d_i, is the number of vertices adjacent to v_i. Throughout this paper, we assume that the vertex degrees are ordered by $d_1 \geq d_2 \geq \cdots \geq d_n$.

The complete graph, the complete bipartite graph and the star of order n are denoted by K_n, $K_{p,q}$ $(p + q = n)$ and S_n, respectively. Let $G - m$ be the graph obtained by deleting any edge m from the graph G and let $\overline{G}$ be the complement of G. Let $G \cup H$ be the vertex-disjoint union of the graphs G and H and let $G \vee H$ be the graph obtained from $G \cup H$ by adding all possible edges from vertices of G to vertices of H, *i.e.*, $G \vee H = \overline{\overline{G} \cup \overline{H}}$ [1].

Let $L(G) = D(G) - A(G)$ be the Laplacian matrix of the graph G, where $A(G)$ and $D(G)$ are the adjacency matrix and the diagonal matrix of the vertex degrees of G, respectively. The normalized Laplacian matrix of G is defined as $\mathcal{L} = D(G)^{-\frac{1}{2}} L(G) D(G)^{-\frac{1}{2}}$, where $D(G)^{-\frac{1}{2}}$ is the matrix which is obtained by taking $(-\frac{1}{2})$-power of each entry of $D(G)$. The Laplacian eigenvalues and the normalized Laplacian eigenvalues of G are the eigenvalues of $L(G)$ and $\mathcal{L}$, respectively. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the Laplacian eigenvalues and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the normalized Laplacian eigenvalues of G. It is well known that $\mu_n = 0$, $\lambda_n = 0$ and the multiplicities of these zero eigenvalues are equal to the number of connected components of G; see [2, 3].

The number of spanning trees (also known as complexity), $t(G)$, of G is given by the following formula in terms of the Laplacian eigenvalues (see [1], p.39):

$$t(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i. \tag{1}$$

It is known that the number of spanning trees of G is also expressed by the normalized Laplacian eigenvalues as follows (see [1], p.49):

$$t(G) = \left(\frac{\prod_{i=1}^{n} d_i}{2e}\right)\prod_{i=1}^{n-1}\lambda_i. \tag{2}$$

Now we list some known upper bounds for $t(G)$.

- Grimmett [4]:

$$t(G) \le \frac{1}{n}\left(\frac{2e}{n-1}\right)^{n-1}. \tag{3}$$

- Grone and Merris [5]:

$$t(G) \le \left(\frac{n}{n-1}\right)^{n-1}\left(\frac{\prod_{i=1}^{n} d_i}{2e}\right). \tag{4}$$

- Nosal [6]: For r-regular graphs,

$$t(G) \le n^{n-2}\left(\frac{r}{n-1}\right)^{n-1}. \tag{5}$$

- Kelmanns ([1], p.222):

$$t(G) \le n^{n-2}\left(1-\frac{2}{n}\right)^{\overline{e}}, \tag{6}$$

where $\overline{e}$ is the number of edges of $\overline{G}$.

- Das [7]:

$$t(G) \le \left(\frac{2e-d_1-1}{n-2}\right)^{n-2}. \tag{7}$$

- Zhang [8]:

$$t(G) \le \bigl(1+(n-2)a\bigr)(1-a)^{n-2}\frac{1}{n}\left(\frac{2e}{n-1}\right)^{n-1}, \tag{8}$$

where $a = (\frac{n(n-1)-2e}{2en(n-2)})^{1/2}$.

- Feng *et al.* [9]:

$$t(G) \le \left(\frac{d_1+1}{n}\right)\left(\frac{2e-d_1-1}{n-2}\right)^{n-2} \tag{9}$$

and

$$t(G) \le \left(\frac{\sum_{i=1}^{n} d_i^2 + 2e - (d_1+1)^2}{n-2}\right)^{\frac{n-2}{2}}. \tag{10}$$

- Li *et al.* [10]:

$$t(G) \leq d_n\left(\frac{2e - d_1 - 1 - d_n}{n-3}\right)^{n-3}. \tag{11}$$

In [4] Grimmett observed that (3) is the generalization of (5). Grone and Merris [5] stated that by the application of arithmetic-geometric mean inequality, (4) leads to (3). In [7] Das indicated that (7) is sharp for S_n or K_n, but (3), (4), (5) and (6) are sharp only for K_n. Li *et al.* [10] pointed out that (11) is sharp for S_n, K_n, $G \cong K_1 \vee (K_1 \cup K_{n-2})$ or $K_n - m$, but (3) is sharp only for K_n, (7) and (9) are sharp for S_n or K_n. In [8, 9] the authors showed that (8) is always better than (3), and (9) is always better than (7) and (10).

This paper is organized as follows. In Section 2, we give some useful lemmas. In Section 3, we obtain some upper bounds for the number of spanning trees of graphs in terms of the number of vertices, the number of edges and the vertex degrees of graphs. We also show that one of these upper bounds is always better than the upper bound (4).

2 Preliminary lemmas

In this section, we give some lemmas which will be used later. Firstly, we introduce an auxiliary quantity of a graph G on the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ as

$$P = 1 + \sqrt{\frac{2}{n(n-1)}\sum_{v_i \sim v_j}\frac{1}{d_i d_j}},$$

where d_i is the degree of the vertex v_i of G.

Lemma 1 [11] *Let G be a graph with n vertices and normalized Laplacian matrix L without isolated vertices. Then*

$$\sum_{i=1}^{n} \lambda_i = \operatorname{tr}(L) = n$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = \operatorname{tr}\left(L^2\right) = n + 2\sum_{v_i \sim v_j}\frac{1}{d_i d_j}.$$

Lemma 2 [3] *Let G be a graph with n vertices and normalized Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$. Then*

$$0 \leq \lambda_i \leq 2.$$

Moreover, $\lambda_1 = 2$ if and only if a connected component of G is bipartite and nontrivial.

Lemma 3 [3] *Let G be a graph with n vertices and normalized Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$. Then*

$$\lambda_1 \geq \frac{n}{n-1}. \tag{12}$$

Moreover, the equality holds in (12) *if and only if G is a complete graph K_n.*

Lemma 4 [12] *Let G be a graph with n vertices and normalized Laplacian eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$. *Then*

$$\lambda_1 \geq P. \tag{13}$$

Moreover, the equality holds in (13) *if and only if G is a complete graph* K_n.

Lemma 5 [12] *The lower bound* (13) *is always better than the lower bound* (12).

Lemma 6 [12] *Let G be a connected graph with* $n > 2$ *vertices. Then* $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1}$ *if and only if* $G \cong K_n$ *or* $G \cong K_{p,q}$.

Lemma 7 [13] *Let G be a graph with n vertices and without isolated vertices. Suppose G has the maximum vertex degree equal to* d_1. *Then*

$$\sum_{v_i \sim v_j} \frac{1}{d_i d_j} \geq \frac{n}{2d_1}. \tag{14}$$

Moreover, the equality holds in (14) *if and only if G is a regular graph.*

Lemma 8 [14] *Let* $x_i > -1$ *for* $1 \leq i \leq n$. *If* $\sum_{i=1}^n x_i = 0$ *and* $\sum_{i=1}^n x_i^2 \geq c^2(1 - n^{-1})$, *then*

$$\sum_{i=1}^{n} \ln(1 + x_i) \leq \ln\bigl(1 + c - cn^{-1}\bigr) + (n-1)\ln\bigl(1 - cn^{-1}\bigr).$$

3 Main results

Now we present the main results of this paper following the ideas in [8] and [9]. Note that P was defined earlier in the previous section.

Theorem 1 *Let G be a graph with n vertices and without isolated vertices. Then*

$$t(G) \leq \bigl(1 + (n-2)b\bigr)(1-b)^{n-2}\left(\frac{n}{n-1}\right)^{n-1}\left(\frac{\prod_{i=1}^n d_i}{2e}\right), \tag{15}$$

where $b = (\frac{n-1-d_1}{n(n-2)d_1})^{1/2}$.

Proof If G is disconnected, then $t(G) = 0$ and (15) follows. Now we assume that G is connected. From (2), we have

$$0 < t(G) = \left(\frac{\prod_{i=1}^n d_i}{2e}\right)\lambda_1 \cdots \lambda_{n-1}$$

since $\lambda_{n-1} > 0$. Let $q = \frac{n}{n-1}$ and $x_i = \frac{\lambda_i}{q} - 1$ for $1 \leq i \leq n-1$. Then $x_i > -1$. Moreover, by Lemma 1 and Lemma 7, we get

$$\sum_{i=1}^{n-1} x_i = \sum_{i=1}^{n-1}\left(\frac{\lambda_i}{q} - 1\right) = 0$$

and

$$\begin{aligned}\sum_{i=1}^{n-1} x_i^2 &= \sum_{i=1}^{n-1}\left(\frac{\lambda_i}{q}-1\right)^2\\ &= (n-1) - \frac{2\sum_{i=1}^{n-1}\lambda_i}{q} + \frac{\sum_{i=1}^{n-1}\lambda_i^2}{q^2}\\ &\geq (n-1) - 2(n-1) + \left(\frac{n-1}{n}\right)^2\left(n+\frac{n}{d_1}\right)\\ &= \frac{(n-1)^2}{nd_1} - \left(\frac{n-1}{n}\right)\\ &= \frac{(n-1)^2(n-1-d_1)}{n(n-2)d_1}\left(1-\frac{1}{n-1}\right)\\ &= \big((n-1)b\big)^2\left(1-\frac{1}{n-1}\right).\end{aligned}$$

Then by Lemma 8, we obtain

$$\prod_{i=1}^{n-1}(1+x_i) \leq \left(1+(n-1)b-\frac{(n-1)b}{n-1}\right)(1-b)^{n-2}.$$

Therefore, we arrive at

$$\prod_{i=1}^{n-1}\lambda_i \leq \big(1+(n-2)b\big)(1-b)^{n-2}\left(\frac{n}{n-1}\right)^{n-1}$$

and

$$t(G) \leq \big(1+(n-2)b\big)(1-b)^{n-2}\left(\frac{n}{n-1}\right)^{n-1}\left(\frac{\prod_{i=1}^{n} d_i}{2e}\right).$$

Hence, the result holds. □

Remark 1 Let $f(b) = (1+(n-2)b)(1-b)^{n-2}$. Then

$$f'(b) = -(n-2)(n-1)b(1-b)^{n-3} \leq 0$$

for $0 \leq b \leq 1$. Therefore, $f(b) \leq f(0) = 1$; see [8]. Hence, we conclude that the upper bound (15) is always better than the upper bound (4). Moreover, if G is the complete graph K_n, then the equality holds in (15).

Theorem 2 *Let G be a connected graph with $n > 2$ vertices. Then*

$$t(G) \leq P\left(\frac{n-P}{n-2}\right)^{n-2}\left(\frac{\prod_{i=1}^{n} d_i}{2e}\right). \tag{16}$$

Moreover, the equality holds in (16) *if and only if G is the complete graph K_n.*

Proof From (2) and Lemma 1, we get

$$\begin{aligned} t(G) &= \left(\frac{\prod_{i=1}^n d_i}{2e}\right)\prod_{i=1}^{n-1}\lambda_i = \left(\frac{\prod_{i=1}^n d_i}{2e}\right)\lambda_1\prod_{i=2}^{n-1}\lambda_i \\ &\le \left(\frac{\prod_{i=1}^n d_i}{2e}\right)\lambda_1\left(\frac{\sum_{i=2}^{n-1}\lambda_i}{n-2}\right)^{n-2} \\ &= \left(\frac{\prod_{i=1}^n d_i}{2e}\right)\lambda_1\left(\frac{\sum_{i=1}^{n-1}\lambda_i-\lambda_1}{n-2}\right)^{n-2} = \left(\frac{\prod_{i=1}^n d_i}{2e}\right)\lambda_1\left(\frac{n-\lambda_1}{n-2}\right)^{n-2}. \end{aligned}$$

For $P \le x \le 2$, let

$$f(x) = x(n-x)^{n-2}.$$

By Lemma 4 and Lemma 5, we have that

$$\lambda_1 \ge P \ge \frac{n}{n-1}$$

and

$$f'(x) = f(x)\frac{n-(n-1)x}{x(n-x)} \le 0$$

for $P \le x \le 2$. Hence, $f(x)$ takes its maximum value at $x = P$ and (16) follows.

If the equality holds in (16), then all inequalities in the above argument must be equalities. Hence, we have

$$\lambda_1 = P \qquad \text{and} \qquad \lambda_2 = \cdots = \lambda_{n-1}.$$

Then by Lemma 4 and Lemma 6, we conclude that G is the complete graph K_n.

Conversely, we can easily see that the equality holds in (16) for the complete graph K_n. □

Now we consider the bipartite graph case of the above theorem.

Theorem 3 *Let G be a connected bipartite graph with n > 2 vertices. Then*

$$t(G) \le \frac{\prod_{i=1}^n d_i}{e}. \tag{17}$$

Moreover, the equality holds in (17) *if and only if* $G \cong K_{p,q}$.

Proof Since G is a connected bipartite graph, by Lemma 2, we have $\lambda_1 = 2$. Considering this, (2) and Lemma 1, we obtain

$$\begin{aligned} t(G) &= \left(\frac{\prod_{i=1}^n d_i}{2e}\right)\prod_{i=1}^{n-1}\lambda_i = \left(\frac{\prod_{i=1}^n d_i}{2e}\right)\lambda_1\prod_{i=2}^{n-1}\lambda_i \\ &\le \left(\frac{\prod_{i=1}^n d_i}{e}\right)\left(\frac{\sum_{i=2}^{n-1}\lambda_i}{n-2}\right)^{n-2} = \left(\frac{\prod_{i=1}^n d_i}{e}\right)\left(\frac{n-\lambda_1}{n-2}\right)^{n-2} = \frac{\prod_{i=1}^n d_i}{e}. \end{aligned}$$

Moreover, the equality holds in (17) if and only if $\lambda_2 = \cdots = \lambda_{n-1}$, by Lemma 6, *i.e.*, if and only if $G \cong K_{p,q}$. □

Competing interests
The author declares that she has no competing interests.

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