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On Absolute Weighted Mean Summability of Orthogonal Series

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Abstract. In this paper we prove two theorems on absolute weighted mean summability of orthogonal series. These theorems generalize results of the paper [4].

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1. Introduction and Preliminaries

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with its partial sums $\{s_n\}$, and let $A = (a_{nv})$ be a normal matrix, that is, lower-semi matrix with nonzero entries. By $(A_n(s))$ we denote the A-transform of the sequence $s = \{s_n\}$, i.e.,

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v.$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|A|_k$, $k \geq 1$, [5] if

$$\sum_{n=0}^{\infty} |a_{nn}|^{1-k} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

In the special case when A is a generalized Nörlund matrix (resp. $k = 1$), $|A|_k$ summability is the same as $|N, p, q|_k$ (resp. $|N, p, q|$) summability [6] (see [3]). By a generalized Nörlund matrix we mean one such that

$$\begin{aligned} a_{nv} &= \frac{p_{n-v} q_v}{R_n} \quad \text{for} \quad 0 \leq v \leq n, \\ a_{nv} &= 0 \quad \text{for} \quad v > n, \end{aligned}$$

where for two given sequences of positive real constants $p = \{p_n\}$ and $q = \{q_n\}$, the convolution $R_n := (p * q)_n$ is defined by

$$(p * q)_n = \sum_{v=0}^{n} p_v q_{n-v} = \sum_{v=0}^{n} p_{n-v} q_v.$$

When $(p * q)_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}(s)\}$ defined by

$$t_n^{p,q}(s) = \frac{1}{R_n} \sum_{m=0}^{n} p_{n-m} q_m s_m$$

and $|A|_k$ summability reduces to the following definition:
The infinite series $\sum_{n=0}^{\infty} a_n$ is absolutely summable $(N, p, q)_k$, $k \geq 1$, if the series

$$\sum_{n=0}^{\infty} \left(\frac{R_n}{q_n} \right)^{k-1} |t_n^{p,q}(s) - t_{n-1}^{p,q}(s)|^k$$

converges (see [6]), and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |N, p, q|_k.$$

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the interval (a, b). We assume that $f(x)$ belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x), \tag{1.1}$$

where $c_n = \int_a^b f(x)\varphi_n(x)dx$, $(n = 0, 1, 2, \dots)$.
We write

$$R_n^j := \sum_{v=j}^{n} p_{n-v} q_v, \;\; R_n^{n+1} = 0, \;\; R_n^0 = R_n$$

and

$$P_n := (p * 1)_n = \sum_{v=0}^{n} p_v \quad \text{and} \quad Q_n := (1 * q)_n = \sum_{v=0}^{n} q_v.$$

Regarding to $|N, p, q| \equiv |N, p, q|_1$ summability of the orthogonal series (1.1) the following two theorems are proved.

Theorem .1.1. [4] *If the series*

$$\sum_{n=0}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |c_j|^2 \right\}^{\frac{1}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $|N, p, q|$ *almost everywhere.*

Theorem 1.2. [4] *Let* $\{\Omega(n)\}$ *be a positive sequence such that* $\{\Omega(n)/n\}$ *is a non-increasing sequence and the series* $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ *converges. Let* $\{p_n\}$ *and* $\{q_n\}$ *be non-negative. If the series* $\sum_{n=1}^{\infty} |c_n|^2 \Omega(n) w^{(1)}(n)$ *converges, then the orthogonal series* $\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |N, p, q|$ *almost everywhere, where* $w^{(1)}(n)$ *is defined by* $w^{(1)}(j) := j^{-1} \sum_{n=j}^{\infty} n^2 \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2$.

The main purpose of this paper is studying of the $|A|_k$ summability of the orthogonal series (1.1), for $1 \le k \le 2$, and to deduce as corollaries all results of Y. Okuyama [4]. Before doing this first introduce some further notations.
Given a normal matrix $A := (a_{nv})$, we associate two lower semi matrices $\bar{A} := (\bar{a}_{nv})$ and $\hat{A} := (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} := \sum_{i=v}^{n} a_{ni}, \ n, i = 0, 1, 2, \ldots$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \ \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \ n = 1, 2, \ldots$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively.
The following lemma due to Beppo Levi (see, for example [7]) is often used in the theory of functions. It will need us to prove main results.

Lemma 1.1. *If* $f_n(t) \in L(E)$ *are non-negative functions and*

$$\sum_{n=1}^{\infty} \int_E f_n(t)dt < \infty, \tag{1.2}$$

then the series

$$\sum_{n=1}^{\infty} f_n(t)dt$$

converges almost everywhere on E *to a function* $f(t) \in L(E)$*. Moreover, the series (1.2) is also convergent to* f *in the norm of* $L(E)$*.*

Throughout this paper K denotes a positive constant that it may depends only on k, and be different in different relations.

2. Main Results

We prove the following theorem.

Theorem 2.1. *If for $1 \leq k \leq 2$ the series*

$$\sum_{n=1}^{\infty} \left\{ |a_{nn}|^{\frac{2}{k}-2} \sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

is summable $|A|_k$ almost everywhere.

Proof. For the matrix transform $A_n(s)(x)$ of the partial sums of the orthogonal series $\sum_{n=0}^{\infty} c_n \varphi_n(x)$ we have

$$\begin{aligned} A_n(s)(x) &= \sum_{v=0}^{n} a_{nv} s_v(x) = \sum_{v=0}^{n} a_{nv} \sum_{j=0}^{v} c_j \varphi_j(x) \\ &= \sum_{j=0}^{n} c_j \varphi_j(x) \sum_{v=j}^{n} a_{nv} = \sum_{j=0}^{n} \bar{a}_{nj} c_j \varphi_j(x) \end{aligned}$$

where $\sum_{j=0}^{v} c_j \varphi_j(x)$ is the partial sum of order v of the series (1.1).
Hence

$$\begin{aligned} \bar{\Delta} A_n(s)(x) &= \sum_{j=0}^{n} \bar{a}_{nj} c_j \varphi_j(x) - \sum_{j=0}^{n-1} \bar{a}_{n-1,j} c_j \varphi_j(x) \\ &= \bar{a}_{nn} c_n \varphi_n(x) + \sum_{j=0}^{n-1} (\bar{a}_{n,j} - \bar{a}_{n-1,j}) c_j \varphi_j(x) \\ &= \hat{a}_{nn} c_n \varphi_n(x) + \sum_{j=0}^{n-1} \hat{a}_{n,j} c_j \varphi_j(x) = \sum_{j=0}^{n} \hat{a}_{n,j} c_j \varphi_j(x). \end{aligned}$$

Using the Hölder's inequality and orthogonality to the latter equality, we have that

$$\begin{aligned} \int_a^b |\bar{\Delta} A_n(s)(x)|^k dx &\leq (b-a)^{1-\frac{k}{2}} \left(\int_a^b |A_n(s)(x) - A_{n-1}(s)(x)|^2 dx \right)^{\frac{k}{2}} \\ &= (b-a)^{1-\frac{k}{2}} \left(\int_a^b \left| \sum_{j=0}^{n} \hat{a}_{n,j} c_j \varphi_j(x) \right|^2 dx \right)^{\frac{k}{2}} \end{aligned}$$

$$= (b-a)^{1-\frac{k}{2}} \left[\sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{\frac{k}{2}}.$$

Thus, the series

$$\sum_{n=1}^{\infty} |a_{nn}|^{1-k} \int_a^b |\bar{\Delta} A_n(s)(x)|^k dx \leq K \sum_{n=1}^{\infty} \left\{ |a_{nn}|^{\frac{2}{k}-2} \sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right\}^{\frac{k}{2}} \tag{2.1}$$

converges by the assumption. From this fact and since the functions $|\bar{\Delta} A_n(s)(x)|$ are non-negative, then by the Lemma 1.1 the series

$$\sum_{n=1}^{\infty} |a_{nn}|^{1-k} |\bar{\Delta} A_n(s)(x)|^k$$

converges almost everywhere. This completes the proof of the theorem.

If we put

$$\mathcal{H}^{(k)}(A;j) := \frac{1}{j^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} n^{\frac{2}{k}} |n a_{nn}|^{\frac{2}{k}-2} |\hat{a}_{n,j}|^2 \tag{2.2}$$

then the following theorem holds true.

Theorem 2.2. *Let $1 \leq k \leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. If the following series $\sum_{n=1}^{\infty} |c_n|^2 \Omega^{\frac{2}{k}-1}(n) H^{(k)}(A;n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |A|_k$ almost everywhere, where $H^{(k)}(A;j)$ is defined by (2.2).*

Proof. Applying Hölder's inequality to the inequality (2.1) we get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} |a_{nn}|^{1-k} \int_a^b |\bar{\Delta} A_n(s)(x)|^k dx \leq \\
&\leq K \sum_{n=1}^{\infty} |a_{nn}|^{1-k} \left[\sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{\frac{k}{2}} \\
&= K \sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))^{\frac{2-k}{2}}} \left[|a_{nn}|^{\frac{2}{k}-2} (n\Omega(n))^{\frac{2}{k}-1} \sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{\frac{k}{2}} \\
&\leq K \left(\sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))} \right)^{\frac{2-k}{2}} \left[\sum_{n=1}^{\infty} |a_{nn}|^{\frac{2}{k}-2} (n\Omega(n))^{\frac{2}{k}-1} \sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{\frac{k}{2}}
\end{aligned}$$

$$\leq K\left\{\sum_{j=1}^{\infty}|c_j|^2\sum_{n=j}^{\infty}|a_{nn}|^{\frac{2}{k}-2}\left(n\Omega(n)\right)^{\frac{2}{k}-1}|\hat{a}_{n,j}|^2\right\}^{\frac{k}{2}}$$

$$\leq K\left\{\sum_{j=1}^{\infty}|c_j|^2\left(\frac{\Omega(j)}{j}\right)^{\frac{2}{k}-1}\sum_{n=j}^{\infty}n^{\frac{2}{k}}|na_{nn}|^{\frac{2}{k}-2}|\hat{a}_{n,j}|^2\right\}^{\frac{k}{2}}$$

$$= K\left\{\sum_{j=1}^{\infty}|c_j|^2\Omega^{\frac{2}{k}-1}(j)\mathcal{H}^{(k)}(A;j)\right\}^{\frac{k}{2}},$$

which is finite by virtue of the hypothesis of the theorem, and this completes the proof of the theorem.

For $a_{n,v}=\frac{p_{n-v}q_v}{R_n}$ we have $a_{n,n}=\frac{p_0q_n}{R_n}$ and

$$\begin{aligned}\hat{a}_{n,v} &= \bar{a}_{n,v}-\bar{a}_{n-1,v}\\ &= \sum_{j=v}^{n}a_{nj}-\sum_{j=v}^{n-1}a_{n-1,j}\\ &= \frac{1}{R_n}\sum_{j=v}^{n}p_{n-j}q_j-\frac{1}{R_{n-1}}\sum_{j=v}^{n-1}p_{n-1-j}q_j\\ &= \frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\end{aligned}$$

therefore the following corollaries follow from the main results:

Corollary 2.1. If for $1\leq k\leq 2$ the series

$$\sum_{n=1}^{\infty}\left\{\left(\frac{R_n}{q_n}\right)^{2-\frac{2}{k}}\sum_{j=0}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2|c_j|^2\right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}c_n\varphi_n(x)$$

is summable $|N,p,q|_k$ almost everywhere.

Corollary 2.2. Let $1\leq k\leq 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ converges. If the following series $\sum_{n=1}^{\infty}|c_n|^2\Omega^{\frac{2}{k}-1}(n)\mathcal{N}^{(k)}(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} c_n\varphi_n(x) \in |N,p,q|_k$ almost everywhere, where $\mathcal{N}^{(k)}(j)$ is defined by

$$\mathcal{N}^{(k)}(j) := \frac{1}{j^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} n^{\frac{4}{k}-2} \left(\frac{R_n}{q_n}\right)^{2-\frac{2}{k}} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}}\right)^2 .$$

Remark 2.1. *We note that for $k = 1$ corollaries 2.1 and 2.2 reduce in theorems 1.1 and 1.2 respectively.*

Let us prove now another two corollaries that follow from the corollary 2.1.

Corollary 2.3. If for $1 \le k \le 2$ the series

$$\sum_{n=0}^{\infty} \left(\frac{p_n}{P_n^{1/k} P_{n-1}}\right)^k \left\{\sum_{j=1}^{n} p_{n-j}^2 \left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\right)^2 |a_j|^2\right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} a_n\varphi_n(x)$$

is summable $|N,p|_k$ almost everywhere.

Proof. After some elementary calculations one can show that

$$\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} = \frac{p_n}{P_n P_{n-1}} \left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\right) p_{n-j}$$

for all $q_n = 1$, and the proof follows immediately from Theorem 2.1.

Corollary 2.4. If for $1 \le k \le 2$ the series

$$\sum_{n=0}^{\infty} \left(\frac{q_n^{1/k}}{Q_n^{1/k} Q_{n-1}}\right)^k \left\{\sum_{j=1}^{n} Q_{j-1}^2 |a_j|^2\right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} a_n\varphi_n(x)$$

is summable $|\overline{N},q|_k$ almost everywhere.

Proof. From the fact that

$$\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} = -\frac{q_n Q_{j-1}}{Q_n Q_{n-1}}$$

for all $p_n = 1$, the proof follows immediately from Theorem 2.1.

Remark 2.2. *For $k = 1$ corollaries 2.3 and 2.4 are proved earlier in [1] and [2].*

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