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ON THE ENERGY AND ESTRADA INDEX OF STRONGLY QUOTIENT GRAPHS

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In this paper, we consider the strongly quotient graphs and obtain some better results for the energy and Estrada index of these graphs, as well as some relations between Estrada index and the graph energy.

Key words : Graph labeling, Energy of graph, Estrada index of graph.

1. Introduction

Let G be a graph with n vertices and m edges and let the vertices of G are labeled as $v_1, v_2, \ldots, v_n$. Let d_i be the degree of ith vertex of G, $i = 1, 2, \ldots, n$. Let $A(G)$ be the $(0, 1)$-adjacency matrix of G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues. The eigenvalues of the adjacency matrix of G are said to be [5] eigenvalues of G

and to form spectrum of G. Since $A(G)$ is a real symmetric matrix its eigenvalues are real numbers. So we can order them so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If G possesses q quadrangles, then

$$\sum_{i=1}^{n} \lambda_i^2 = 2m$$

and

$$\sum_{i=1}^{n} \lambda_i^4 = 2\sum_{i=1}^{n} d_i^2 - 2m + 8q.$$

The energy of the graph G is defined by

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| . \tag{1}$$

It is closely related to the total π-electron energy in a molecule represented by a (molecular) graph. An extensive work has been done on $E(G)$ and it can be found in the literature. For detailed information see [12-14].

During the past forty years or so an enormous amount of research work has been done on graph labeling, where the vertices are asigned values subject to certain conditions. These interesting problems have been motivated by practical problems. Recently, Adiga *et al*. [3] have introduced the notion of strongly quotient graphs and studied these type of graphs. Throughout this paper by a labeling f of a graph G of order n we mean an injective mapping

$$f : V(G) \to \{1, 2, \ldots, n\} .$$

We define the quotient function

$$f_q : E_G \to Q$$

by

$$f_q(e) = \min\left\{\frac{f(v)}{f(w)}, \frac{f(w)}{f(v)}\right\}$$

if e joins v and w. Note that for any $e \in E_G$, $0 < f_q(e) < 1$.

A graph with n vertices called a strongly quotient graph if its vertices can be labeled $1, 2, \ldots, n$ such that the quotient function f_q is injective i.e. the values $f_q\left(e\right)$ on the edges are all distinct. For survey and detailed information for graph labeling, strongly quotient graphs, graph spectra, energy and Estrada index of graphs see [2-5, 12-18]. Throughout this paper SQG stands for strongly quotient graph of order n with maximum number of edges.

We organized this paper in the following way. In Section 2, we obtain a better upper bound and two new lower bounds for the energy of SQG. In Section 3, we study on the Estrada index of SQG and obtain some better lower and upper bounds for it involving graph energy and several other graph invariants. Now we give some lemmas which will be needed then.

Lemma 1.1. [1] — Let $a_1, a_2, \ldots, a_n$ be non-negative numbers. Then

$$\begin{aligned} n\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{1/n}\right] &\leq n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2 \\ &\leq n\left(n-1\right)\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{1/n}\right]. \end{aligned}$$

Lemma 1.2. [4] — If G is a SQG then -1 is an eigenvalue of G and its multiplicity is greater than or equal to $|P| = l$ where

$$P = \left\{p \mid \ p \text{ is prime and } \frac{n}{2} < p \leq n\right\}.$$

Lemma 1.3. [4] — If G is a strongly quotient graph (SQG) with n vertices then 0 is an eigenvalue of G with multiplicity greater than or equal to s where

$$s = \sum_{\substack{p\text{-prime} \\ p \leq \left[\frac{n}{2}\right]}} \left(\left[\log_p n\right] - 1\right).$$

2. On the Bounds for Energy of SQG

In this section, we will present some upper and lower bounds for $E(G)$ where G is SQG, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let n_+ be the number of positive eigenvalues of G and l and s are as defined in Lemmas 1.2 and 1.3 respectively. For our convenience we rename the eigenvalues such that $\lambda_{n-l-s+1} = \lambda_{n-l-s+2} = \cdots = \lambda_{n-l} = 0$, $\lambda_{n-l+1} = \lambda_{n-l+2} = \cdots = \lambda_n = -1$.

***Theorem* 2.1** — *Let G be a strongly quotient graph (SQG) with $n > 3$ vertices and maximum edges m. Let $P = \left\{p : p \text{ is prime and } \frac{n}{2} < p \leq n\right\}$ and $|P| = l$. Then*

$$E(G) \geq l + \sqrt{2m - l + (n-l-s)(n-l-s-1)\Delta^{2/n-l-s}} \tag{2}$$

and

$$E(G) \leq l + \sqrt{(n-l-s-1)(2m-l) + (n-l-s)\Delta^{2/n-l-s}} \tag{3}$$

where

$$\Delta = \prod_{i=1}^{n-l-s} |\lambda_i| \text{ and } s = \sum_{\substack{p\text{-prime} \\ p \leq \left[\frac{n}{2}\right]}} \left([\log_p n] - 1\right).$$

Proof : Setting $a_i = \lambda_i^2$ and replacing n by $n-l-s$ in Lemma 1.1 we obtain

$$T \leq (n-l-s)\sum_{i=1}^{n-l-s} \lambda_i^2 - \left(\sum_{i=1}^{n-l-s} |\lambda_i|\right)^2 \leq (n-l-s-1)\,T$$

where

$$T = (n-l-s)\left[\frac{1}{n-l-s}\sum_{i=1}^{n-l-s} \lambda_i^2 - \left(\prod_{i=1}^{n-l-s} \lambda_i^2\right)^{1/n-l-s}\right].$$

By Lemmas 1.2 and 1.3, we have -1 and 0 are eigenvalues of G with multiplicity greater than or equal to l and s, respectively. Therefore we obtain

$$T \leq (n-l-s)(2m-l) - (E(G)-l)^2 \leq (n-l-s-1)\,T.$$

Observe that

$$\begin{aligned} T &= (n-l-s)\left[\frac{1}{n-l-s}\sum_{i=1}^{n-l-s}\lambda_i^2-\left(\prod_{i=1}^{n-l-s}\lambda_i^2\right)^{1/n-l-s}\right] \\ &= (n-l-s)\left[\frac{1}{n-l-s}(2m-l)-\left(\prod_{i=1}^{n-l-s}|\lambda_i|\right)^{2/n-l-s}\right] \\ &= 2m-l-(n-l-s)\,\Delta^{2/n-l-s}. \end{aligned}$$

Hence we get the result.

Remark 2.2 : In [4] Adiga and Zaferani determined two eigenvalues of SQG and also obtained the following upper bound for the energy of SQG

$$E(G) \le l + \sqrt{(n-l-s)(2m-l)}. \tag{4}$$

The upper bound (3) is sharper than the upper bound (4). Using arithmetic-geometric mean inequality, we obtain

$$2m-l \ge (n-l-s)\,\Delta^{2/n-l-s}$$

and considering the upper bound (3) we arrive at

$$E(G) \le l + \sqrt{(n-l-s)(2m-l)}$$

which is the upper bound (4).

***Theorem* 2.3.** — *Let G be a strongly quotient graph (SQG) with $n > 3$ vertices and maximum edges m. Let $P = \left\{p :\ p \text{ is prime and } \frac{n}{2} < p \le n\right\}$ and $|P| = l$. Then*

$$E(G) \ge l + (2m-l)\sqrt{\frac{2m-l}{2\sum_{i=1}^{n} d_i^2 - 2m + 8q - l}}. \tag{5}$$

PROOF : We start with the Hölder inequality

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{1/p}\left(\sum_{i=1}^{n} b_i^q\right)^{1/q} \tag{6}$$

which holds for non-negative real numbers $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$. Setting $a_i = |\lambda_i|^{2/3}$, $b_i = |\lambda_i|^{4/3}$, $p = 3/2$, $q = 3$ and replacing n by $n - l - s$ in (6), we obtain

$$\sum_{i=1}^{n-l-s} |\lambda_i|^2 = \sum_{i=1}^{n-l-s} |\lambda_i|^{2/3} \left(|\lambda_i|^4\right)^{1/3} \leq \left(\sum_{i=1}^{n-l-s} |\lambda_i|\right)^{2/3} \left(\sum_{i=1}^{n-l-s} |\lambda_i|^4\right)^{1/3}. \tag{7}$$

Since G has $n > 3$ vertices, the cardinality of P is equal to at least one. Then $\sum\limits_{i=1}^{n-l-s} |\lambda_i|^4 \neq 0$ and (7) can be written as the following

$$\sum_{i=1}^{n-l-s} |\lambda_i| \geq \sqrt{\frac{\left(\sum\limits_{i=1}^{n-l-s} |\lambda_i|^2\right)^3}{\sum\limits_{i=1}^{n-l-s} |\lambda_i|^4}} = \sqrt{\frac{\left(\sum\limits_{i=1}^{n-l-s} \lambda_i^2\right)^3}{\sum\limits_{i=1}^{n-l-s} \lambda_i^4}}.$$

By Lemma 1.2 and 1.3 we have -1 and 0 are eigenvalues of G with multiplicity greater than or equal to l and s, respectively. Therefore we obtain

$$E(G) - l \geq \sqrt{\frac{(2m-l)^3}{2\sum\limits_{i=1}^{n} d_i^2 - 2m + 8q - l}}$$

or equivalently

$$E(G) \geq l + (2m-l)\sqrt{\frac{2m-l}{2\sum\limits_{i=1}^{n} d_i^2 - 2m + 8q - l}}.$$

Hence the result.

3. On the Estrada Index of SQG

In this section, we first recall that Estrada index of the graph G is defined by

$$EE = EE(G) = \sum_{i=1}^{n} e^{\lambda_i}. \tag{8}$$

The k-th moment of the graph G, denoted by $M_k = M_k(G)$, is given by

$$M_k = M_k(G) = \sum_{i=1}^{n} (\lambda_i)^k .$$

It is well known that [5] $M_k(G)$ is equal to the number of self-returning walks of length k of the graph G.

Recalling the power series expansion of e^x we have another expression of Estrada index as

$$EE = \sum_{k=0}^{\infty} \frac{M_k}{k!}. \tag{9}$$

The Estrada index of graphs has found a remarkable variety of applications in Chemistry and Physics. In addition, there exist a vast literature that studies Estrada index and its lower and upper bounds. For survey and more information we refer to the reader [6-11, 15-18].

Now we will give some better results for $EE(G)$ involving graph energy $E(G)$ and several other graph invariants where G is SQG.

***Theorem* 3.1.** — *The Estrada index $EE(G)$ and the graph energy $E(G)$ of SQG with $n > 3$ vertices and maximum edges m satisfy the following inequality*

$$\begin{aligned} le^{-1} &+ \frac{1}{2}E(G)(e-1) + n - n_+ \leq EE(G) \\ &\leq le^{-1} + (n-l-1) + e^{\frac{E(G)}{2}} \end{aligned} \tag{10}$$

where $P = \left\{p : \ p \text{ is prime and } \frac{n}{2} < p \leq n\right\}$ *and* $|P| = l$.

PROOF : Lower bound: Considering Lemma 1.2 and 1.3 and the inequalities $e^x \geq xe$ and $e^x \geq 1+x$, we get

$$\begin{aligned} EE(G) &= \sum_{i=1}^{n} e^{\lambda_i} = se^0 + le^{-1} + \sum_{i=1}^{n-l-s} e^{\lambda_i} \\ &\geq s + le^{-1} + \sum_{\lambda_i>0} e\lambda_i + \sum_{\substack{i=1 \\ \lambda_i \leq 0}}^{n-l-s} (1+\lambda_i) \\ &= s + le^{-1} + (e-1)\left(\lambda_1 + \lambda_2 + \cdots + \lambda_{n_+}\right) + (n-l-s-n_+) + l \\ &= le^{-1} + \frac{1}{2}E(G)(e-1) + n - n_+. \end{aligned}$$

Hence the lower bound in (10).

Upper Bound : Let n_+ be the number of positive eigenvalues of G. Since $f(x) = e^x$ monotonically increases in the interval $(-\infty, +\infty)$, we obtain

$$\begin{aligned} EE(G) &= \sum_{i=1}^{n} e^{\lambda_i} = se^0 + le^{-1} + \sum_{i=1}^{n-l-s} e^{\lambda_i} \\ &\leq s + le^{-1} + (n-l-s-n_+) + \sum_{i=1}^{n_+} e^{\lambda_i} \end{aligned}$$

Therefore

$$\begin{aligned} EE(G) &\leq le^{-1} + (n-l-n_+) + \sum_{i=1}^{n_+}\sum_{k\geq 0} \frac{(\lambda_i)^k}{k!} \\ &= le^{-1} + n - l + \sum_{k\geq 1} \frac{1}{k!} \sum_{i=1}^{n_+} (\lambda_i)^k \qquad (11) \\ &\leq le^{-1} + n - l + \sum_{k\geq 1} \frac{1}{k!} \left[\sum_{i=1}^{n_+} (\lambda_i)\right]^k \\ &= le^{-1} + (n-l-1) + e^{\frac{E(G)}{2}}. \end{aligned}$$

This completes the proof.

Remark 3.2. : In [17] Liu and Liu proved that the following inequality for simple graphs with n vertices and m edges

$$\frac{1}{2}E\left(G\right)\left(e-1\right)+n-n_{+}\leq EE\left(G\right)\leq n-1+e^{\frac{E(G)}{2}}. \qquad (12)$$

It is clear that the lower bound in (10) is better than the lower bound in (12) for $EE\left(G\right)$ of SQG with $n>3$ vertices and maximum edges m. Since the function $f\left(x\right)=e^{x}$ monotonically increases in the interval $\left(-\infty,+\infty\right)$ and considering the cardinality of P $\left(\left|P\right|=l\right)$ is equal to at least one, we also conclude that the upper bound in (10) is better than the upper bound in (12).

***Theorem* 3.3.** — *The Estrada index $EE\left(G\right)$ and the graph energy $E\left(G\right)$ of SQG with $n>3$ vertices and maximum edges m satisfy the following inequality*

$$EE\left(G\right)-E\left(G\right)\leq le^{-1}+\left(n-l-1\right)-\sqrt{2m-l}+e^{\sqrt{2m-l}}. \qquad (13)$$

where $P=\left\{p:\ p\ \text{is prime and}\ \frac{n}{2}<p\leq n\right\}$ *and* $\left|P\right|=l$.

PROOF : From (11)

$$\begin{aligned} EE\left(G\right) &\leq le^{-1}+n-l+\sum_{k\geq1}\frac{1}{k!}\sum_{i=1}^{n_{+}}\left(\lambda_{i}\right)^{k} \\ &= le^{-1}+n-l+\frac{E\left(G\right)}{2}+\sum_{k\geq2}\frac{1}{k!}\sum_{i=1}^{n_{+}}\left(\lambda_{i}\right)^{k} \\ &< le^{-1}+n-l+E\left(G\right)+\sum_{k\geq2}\frac{1}{k!}\sum_{i=1}^{n_{+}}\left(\lambda_{i}\right)^{k} \end{aligned}$$

Therefore we get

$$\begin{aligned} EE\left(G\right) &\leq le^{-1}+n-l+E\left(G\right)+\sum_{k\geq2}\frac{1}{k!}\left[\sum_{i=1}^{n_{+}}\left(\lambda_{i}\right)^{2}\right]^{\frac{k}{2}} \\ &= le^{-1}+n-l+E\left(G\right)+\sum_{k\geq2}\frac{1}{k!}\left[2m-\sum_{i=n_{+}+1}^{n}\left(\lambda_{i}\right)^{2}\right]^{\frac{k}{2}}. \end{aligned}$$

By Lemma 1.2 and 1.3, we have -1 and 0 are eigenvalues of the strongly quotient graph G with multiplicity greater than or equal to l and s, respectively. These imply that

$$\sum_{i=n_{+}+1}^{n} (\lambda_i)^2 \geq l.$$

Hence we obtain

$$\begin{aligned} EE(G) - E(G) &\leq le^{-1} + n - l + \sum_{k\geq 2} \frac{1}{k!} [2m-l]^{\frac{k}{2}} \\ &= le^{-1} + (n-l-1) - \sqrt{2m-l} + e^{\sqrt{2m-l}}. \end{aligned}$$

From the above discussions we also have

$$EE(G) - \frac{E(G)}{2} \leq le^{-1} + (n-l-1) - \sqrt{2m-l} + e^{\sqrt{2m-l}}$$

that is better than the upper bound (13)

Remark 3.4 : In [17] Liu and Liu proved that the following inequality for simple graphs with n vertices and m edges

$$EE(G) - E(G) \leq n - 1 - \sqrt{2m-1} + e^{\sqrt{2m-1}}. \tag{14}$$

Since the functions $f(x) = e^x$ and $f(x) = e^x - x$ monotonically increase in the intervals $(-\infty, +\infty)$ and $(0, +\infty)$, respectively and considering the cardinality of P $(|P| = l)$ is equal to at least one, we conclude that the upper bound (13) is better than the upper bound (14) for SQG.

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REFERENCES

1. B. Zhou, I. Gutman and T. Aleksić, A note on the Laplacian energy of graphs, *MATCH Commun. Math. Comput. Chem.*, **60** (2008), 441-446.

2. C. Adiga and R. K. Zaferani, On coloring of strongly multiplicative and strongly quotient graphs, *Adv. Stud. Cont. Math.*, **13**(2) (2006), 171-182.

3. C. Adiga, R. K. Zaferani and M. Smitha, Strongly quotient graphs, *South East Asian J. Math. & Math. Sc.*, **15**(2) (2007), 119-127.

4. C. Adiga and R. K. Zaferani, Upper bounds for energy of a graph, *Adv. Stud. Cont. Math.*, **16**(2) (2008), 279-285.

5. D. Cvetkovic, M. Doob and H. Sachs, *Spectra of Graphs-Theory and Application* (Third ed. Johann Ambrosius Bart Verlag, Heidelberg, Leipzig, 1995).

6. E. Estrada, Characterization of 3D molecular structure, *Chem. Phys. Lett.* **319** (2000), 713-718.

7. E. Estrada, Characterization of the folding degree of proteins, *Bioinformatics* **18** (2002), 697-704.

8. E. Estrada, Characterization of amino acid contribution to the folding degree of proteins, *Proteins* **54** (2004), 727-737.

9. E. Estrada and J. A. Rodríguez-Velázguez, Subgraph centrality in complex networks, *Phys. Rev.*, **E71** (2005), 056103-1–056103–6.

10. E. Estrada and J. A. Rodríguez-Velázguez, Spectral measures of bipartivity in complex networks, *Phys. Rev.*, **E72** (2005), 046105-1–046105-6.

11. E. Estrada, J.A. Rodríguez-Velázguez and M. Randić, Atomic Branching in molecules, *Int. J.Quantum Chem.*, **106** (2006), 823-832.

12. I. Gutman, The energy of a graph, *Ber. Math. Stat. Sckt. Forschungzentrum Graz*, **103** (1978), 1-22.

13. I. Gutman Acyclic conjugated molecules, trees and their energies, *J. Math. Chem.*, **1** (1987), 123-143.

14. I. Gutman Total π-electron energy of benezoid hydrocarbons, *Topics Curr. Chem.*, **162** (1992) 26-63.

15. I. Gutman, Lower bounds for Estrada index, *Publications de l'Institut Mathématique*, **83**(97) (2008), 1-7.

16. J. A. De la Peña, I. Gutman and J. Rada, Estimating the Estrada index, *Linear Algebra Appl*., **427** (2007), 70-76.

17. J. P. Liu and B. L. Liu, Bounds of the Estrada index of graphs, *Appl. Math. J. Chinese Univ*., **25**(3) (2010), 325-330.

18. Y. Ginosar, I. Gutman, T. Mansour and M. Schork, Estrada index and Chbeyshev polynomials, Chemical Physics Letters, **454**(4-6) (2008), 145-147.