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On Harmonic Curvatures of Null Generalized Helices in $\mathbb{L}^4$

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Abstract. In this study; we give a relation between harmonic curvatures and the Frenet equations of a null curve in a 4−dimensional Lorentz space. Also, we obtain some theorems and we give an example of a null helix.

Key words: Null curve; Harmonic curvature; Null Frenet curve of osculating order 4; Null helix.
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1. Introduction

Let $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ be non-zero vectors in Minkowski 4−space $\mathbb{R}^4_1$. We denoted $\mathbb{R}^4_1$ by $\mathbb{L}^4$. For $x,\ y \in \mathbb{L}^4$

$$\langle x, y \rangle = -x_1 y_1 + \sum_{i=2}^{4} x_i y_i$$

is called *Lorentzian inner product*. The couple $\{\mathbb{R}^4_1, \langle , \rangle\}$ is called *Lorentzian space* and denoted by $\mathbb{L}^4$. Then the vector v of $\mathbb{L}^4$ is called **i)** time-like if $\langle v, v \rangle < 0$, **ii)** space-like if $\langle v, v \rangle > 0$ or $v = 0$, **iii)** null (or light-like) vector if $\langle v, v \rangle = 0, v \neq 0$. An arbitrary curve $\alpha = \alpha(t)$ in $\mathbb{L}^4$ can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\alpha^{'}(t)$ are respectively space-like, time-like or null, [10].

2. Basic Definitions

Definition 1. Let $\alpha : I \longrightarrow \mathbb{L}^4$ be a null curve in $\mathbb{L}^4$. The curve α is called a Frenet curve of osculating order 4, if its 4^{th} order derivatives $\alpha^{'}(t)$, $\alpha^{''}(t)$, $\alpha^{'''}(t)$, $\alpha^{iv}(t)$ are linearly independent and $\alpha^{'}(t)$, $\alpha^{''}(t)$, $\alpha^{'''}(t)$, $\alpha^{iv}(t)$, $\alpha^{v}(t)$ are no longer linearly independent for all $t \in I$. For each null Frenet curve of osculating order 4, one can associate an orthonormal $4-$frame $\{T, N, W_1, W_2\}$ along α (such that $\alpha^{'}(t) = T$) called the Frenet frame and functions $\{k_1, k_2, k_3, k_4\}$ called the Frenet curvatures. Thus from [1], the Frenet equations of a null curve in a 4-dimensional Lorentz manifold are written down as follows:

$$\begin{cases} \nabla_T T = hT + k_1 W_1 \\ \nabla_T N = -hN + k_2 W_1 + k_3 W_2 \\ \nabla_T W_1 = -k_2 T - k_1 N + k_4 W_2 \\ \nabla_T W_2 = -k_3 T - k_4 W_1, \end{cases}$$

where ∇ is the Levi-Civita connection of $\mathbb{L}^4$, h and $\{k_1, k_2, k_3, k_4\}$ are differential functions, T and N are null vectors , W_1 and W_2 are space-like vectors. In these equations, by changing a suitable parameter t, we may take $h = 0$ and other equations stay unchanged. This parameter is called distinguished parameter of the curve [1]. That is,

$$\begin{cases} \nabla_T T = k_1 W_1, \\ \nabla_T N = k_2 W_1 + k_3 W_2, \\ \nabla_T W_1 = -k_2 T - k_1 N + k_4 W_2, \\ \nabla_T W_2 = -k_3 T - k_4 W_1. \end{cases} \tag{1}$$

From [1] again, since T and N are null vectors, W_i , $1 \leq i \leq 2$, are space-like vectors then we have

(2)

$$\begin{cases} \langle T, T\rangle = \langle N, N\rangle = \langle T, W_1\rangle = \langle T, W_2\rangle = \langle N, W_1\rangle = \langle N, W_2\rangle = \langle W_1, W_2\rangle = 0, \\ \langle T, N\rangle = 1 \text{ and } \langle W_1, W_1\rangle = \langle W_2, W_2\rangle = 1. \end{cases}$$

Definition 2. If a null curve $\alpha : I \longrightarrow \mathbb{L}^4$ is a null Frenet curve of osculating order 4 and Frenet curvatures k_i, $1 \leq i \leq 4$ are non-zero constants, then α is called a null $W-curve$ of rank 4.

3. Null Generalized Helix in $\mathbf{L}^4$

Definition 3. [**2**] Assume that $\alpha \subset \mathbb{L}^4$ is a null generalized helix given by curvature functions k_1, k_2, k_3. Then the harmonic curvatures of α in $\mathbb{L}^4$ are written down as follows:

$$H_i = \begin{cases} -\dfrac{k_2}{k_1} & , \quad i = 1 \\ \dfrac{H_1^{'}}{k_3} & , \quad i = 2. \end{cases} \tag{3}$$

Definition 4. Let α be a time-like curve in $\mathbb{L}^4$ with $\alpha^{'}(s) = V_1$. Let $X \epsilon \chi \left(\mathbb{L}^4\right)$ be a constant unit vector field. If

$$\langle V_1, X\rangle = \cosh \varphi \quad \text{(constant)},$$

then α is called a general helix (inclined curve) in $\mathbb{L}^4$, φ is called the slope angle and the space $Sp\{X\}$ is called slope axis [9].

Definition 5. **[2]** A null curve $\alpha : I \longrightarrow \mathbb{L}^4$ is said to be a generalized helix if there exist a non-zero unit constant vector X such that $\left\langle \alpha^{'}(t), X \right\rangle \neq 0$, is $cons\tan t$. Then $Sp\{X\}$ is called slope axis and for the Frenet frame $\{T, N, W_1, W_2\}$ we have

$$\langle N, X\rangle = H_1 \langle T, X\rangle, \langle W_1, X\rangle = 0, \langle W_2, X\rangle = H_2 \langle T, X\rangle .$$

Theorem 1. Let α be a null Frenet curve of osculating order 4 in $\mathbb{L}^4$. Then

$$k_4 = \frac{k_2 + k_1 H_1}{H_2}, \tag{4}$$

where k_1, k_2, k_4 are the Frenet curvatures and H_1, H_2 are harmonic curvatures of α.

Proof. By the use of a previous definition we obtain

$$\begin{aligned} \langle \nabla_T W_1, X\rangle &= 0 \Rightarrow \langle -k_2 T - k_1 N + k_4 W_2, X\rangle = 0 \\ &\Rightarrow -k_2 \langle T, X\rangle - k_1 \langle N, X\rangle + k_4 \langle W_2, X\rangle = 0 \\ &\Rightarrow -k_2 \langle T, X\rangle - k_1 H_1 \langle T, X\rangle + k_4 H_2 \langle T, X\rangle = 0 \\ &\Rightarrow \langle T, X\rangle (-k_2 - k_1 H_1 + k_4 H_2) = 0, \end{aligned}$$

where, since we know that $\langle T, X\rangle \neq 0$, is $cons\tan t$,we can write

$$-k_2 - k_1 H_1 + k_4 H_2 = 0.$$

Thus, we obtain

$$k_4 = \frac{k_2 + k_1 H_1}{H_2}.$$

Theorem 2. Let α be a null Frenet curve of osculating order 4 in $\mathbb{L}^4$. Then

$$\begin{cases} \nabla_T T = k_1 W_1 \\ \nabla_T N = -k_1 H_1 W_1 + \dfrac{H_1^{'}}{H_2} W_2 \\ \nabla_T W_1 = k_1 H_1 T - k_1 N + \dfrac{k_2 + k_1 H_1}{H_2} W_2 \\ \nabla_T W_2 = -\dfrac{H_1^{'}}{H_2} T - \dfrac{k_2 + k_1 H_1}{H_2} W_1, \end{cases}$$

where k_1, k_2 are Frenet curvatures and H_1, H_2 are harmonic curvatures of α, ∇ is the Levi-Civita connection of $\mathbb{L}^4$.

Proof. By using; (1),(3) and (4), we obtain the proof of the theorem.

Theorem 3. Let $\alpha : I \longrightarrow \mathbb{L}^4$ be a null curve in $\mathbb{L}^4$. Then

$$\begin{cases} \langle \nabla_T W_1, W_1 \rangle = \langle \nabla_T T, W_2 \rangle = \langle \nabla_T W_2, W_2 \rangle = 0, \\ \langle \nabla_T W_2, W_1 \rangle = -\langle \nabla_T W_1, W_2 \rangle, \\ \langle \nabla_T N, W_2 \rangle = \dfrac{H_1^{'}}{H_2}, \\ \langle \nabla_T N, W_1 \rangle = -H_1 k_1, \\ \langle \nabla_T T, W_1 \rangle = -\dfrac{k_2}{H_1}. \end{cases}$$

Here; T and N are null vectors , W_1 and W_2 are space-like vectors, $H_1$and H_2 are harmonic curvatures of α, ∇ is the Levi-Civita connection of $\mathbb{L}^4$ and k_1, k_2 are Frenet curvatures of α.

Proof. By using; (1), (2) and (3), we obtain the proof of the theorem.

4. Example

Example 1. Let $\alpha : I \longrightarrow \mathbb{L}^4$ be the null curve defined by

$$\alpha(t) = (\sinh t, \cosh t, -t, 0)$$

and $X = (0, 0, 1, 0)$ be a unit constant vector field in $\mathbb{L}^4$. The tangent vector of α is

$$T = \alpha^{'}(t) = (\cosh t, \sinh t, -1, 0)$$

and $\langle T, T \rangle = 0$, so α is a null curve in $\mathbb{L}^4$. Also, $\langle T, X \rangle = -1 =$ constant. Therefore the curve α is a null helix.

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