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A New Homotopy Analysis Method for Finding the Exact Solution of Systems of Partial Differential Equations

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Abstract. In this paper, the application of a new homotopy analysis method presented for obtaining solutions of systems of non-linear partial differential equations. Theoretical considerations are discussed. To explain the capability and reliability of the new method some examples are provided. The results show that the new technique is very effective and convenient and comparison of the obtained solutions of this new method with those of applying homotopy analysis method have high accuracy.

Key words: New homotopy analysis method; System of partial differential equations.
2000 Mathematics Subject Classification: 65N99.

1. Introduction

In the last two decades with the rapid development of nonlinear science, there has appeared ever increasing interest of scientists and engineers in the analytical techniques for nonlinear problems. The purpose of this paper is to use new homotopy analysis methd that briefly is called NHAM, to a system of differential equations that arise in many areas of mathematics, engineering and physical sciences and illustrate the advantages and simplicity of NHAM as compared to HAM. These equations are often too complicated to be solved exactly and even if an exact solution is obtained, the required calculations may be too complicated. Most scientific problems and phenomena occur nonlinearly. Except a

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limited number of these problems, most of them do not have precise analytical solution, thus we have to use various approximate analytical methods. Very recently, many powerful methods have been presented, such as the Adomian decomposition method (ADM) [8, 18], the variational iteration method (VIM) [9, 10, 15], the homotopy perturbation method (HPM) [5, 7, 12, 16, 17], the differential transform method [3, 6], the homotopy analysis method (HAM) [1, 13, 14] and the others [11]. At first, we explain the new modification of HAM that is called NHAM for finding the exact solution of systems of partial differential equations is presented. The applicability of the new method is verified by the numerical results that obtained of implement method for three examples that are shown in section 3, and also comparisons between this method and HAM are illustrated in this section. The conclusions appear in section 4.

2. Basic Idea of NHAM

To illustrate of basic idea of NHAM, We consider the general form of a system of PDEs can be considered as the following [4]:

$$(1) \qquad \frac{\partial u_i}{\partial t} + N_i\left(x_1, x_2, \cdots, x_{n-1}, t, u_1, \cdots, u_n\right) = g_i\left(x_1, x_2, \cdots, x_{n-1}, t\right),$$

with the following initial conditions:

$$u_i\left(x_1, x_2, ..., x_{n-1}, t_0\right) = f_i\left(x_1, x_2, ..., x_{n-1}\right), \qquad i = 1, \cdots, n,$$

where $N_1, \cdots, N_n$ are non-linear operators, which usually depend on the unknown functions u_i and their derivatives, x_j for $j = 1, ..., n-1$ denotes independent variable and $g_1,\ g_2, \cdots,\ g_n$ are inhomogeneous terms. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. For solving Eq.(1) by means of NHAM we constructs the so-called zero-order deformation equation

$$(2)\ \ (1-q)\left[\frac{\partial \phi_i}{\partial t} - u_{i,0}(r,t)\right] - q\,\hbar_i\,\mathcal{H}_i\left[\frac{\partial \phi_i}{\partial t} + N_i\left(r, t, \phi_1, ..., \phi_n\right) - g_i\left(r, t\right)\right] = 0,$$

where $r = (x_1, ..., x_{n-1})$, $q \in [0, 1]$ is the embedding parameter, $\hbar_i \neq 0$ is a nonzero auxiliary parameter, $\mathcal{H}_i(r, t) \neq 0$ is an auxiliary function, $u_{i,0}(x_1, ..., x_{n-1}, t)$ is an initial guess of $u_i(r, t)$, $\phi_i(r, t; q)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in NHAM. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi_i(r, t; 0) = \int_{t_0}^{t} u_{i,0}(r, t_1)dt_1,$$

$$\phi_i(r, t; 1) = u_i(r, t).$$

Thus, as q increases from 0 to 1, the solution $\phi_i(r, t; q)$ varies from

$$\int_{t_0}^{t} u_{i,0}(r, t_1)dt_1,$$

to the solution $u_i(r,t)$. Expanding $\phi_i(r,t;q)$ in Taylor series with respect to q, we have

$$\phi_i(r,t;q) = u_{i,0}(r,t) + \sum_{m=1}^{+\infty} u_{i,m}(r,t)q^m, \tag{3}$$

where

$$u_{i,m}(r,t) = \frac{1}{m!}\frac{\partial^m \phi_i(r,t;q)}{\partial q^m}|_{q=0}. \tag{4}$$

If the initial guess, the auxiliary parameter $\hbar_i$, and the auxiliary function are so properly chosen, the series (3) converges at $q = 1$, then we have

$$u_i(r,t) = u_{i,0}(r,t) + \sum_{m=1}^{+\infty} u_{i,m}(r,t), \tag{5}$$

which must be one of solutions of original nonlinear equation, as proved by [2]. As $\hbar_i = -1$ and $\mathcal{H}_i(x_1, x_2, ..., x_{n-1}, t) = 1$, Eq.(2) becomes

$$(1-q)[\frac{\partial \phi_i(r,t;q)}{\partial t} - u_{i,0}(r,t)] + q\,[\frac{\partial \phi_i}{\partial t} + N_i\,(r,t,\phi_1,...,\phi_n) - g_i\,(r,t)] = 0, \tag{6}$$

which is used mostly in the new homotopy perturbation method [12], where as the solution obtained directly, without using Taylor series [5, 12]. According to the definition, the governing equation can be deduced from the zero-order deformation Eq.(2). Define the vector

$$\overrightarrow{u}_{i,n}(r,t) = \{u_{i,0}(r,t), u_{i,1}(r,t), \ldots, u_{i,n}(r,t)\}.$$

Differentiating Eq.(2) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\frac{\partial}{\partial t}[u_{i,m}(r,t) - \chi_m u_{i,m-1}(r,t)] = \hbar_i\,\mathcal{H}_i(r,t)\mathcal{R}_{i,m}(\overrightarrow{u}_{1,m-1}(r,t), \ldots, \overrightarrow{u}_{n,m-1}(r,t)), \tag{7}$$

where

$$\mathcal{R}_{i,m}(\overrightarrow{u}_{1,m-1}, \ldots, \overrightarrow{u}_{n,m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\left[\frac{\partial \phi_i}{\partial t} + N_i\,(r,t,\phi_1,...,\phi_n) - g_i\right]}{\partial q^{m-1}}|_{q=0},$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{8}$$

Therefore, the solution of Eq.(1), can be readily obtained by

$$u_i(r,t) = \sum_{j=0}^{+\infty} u_{i,j}(r,t).$$

In practice, all terms of series $u_i(r,t) = \sum_{j=0}^{+\infty} u_{i,j}(r,t)$ cannot be determined and so we use an approximation of the solution by the following truncated series

$$\mu_{i,m}(r,t) = \sum_{j=0}^{m-1} u_{i,j}(r,t),$$

where

$$u_i(r,t) = \lim_{m \to +\infty} \mu_{i,m}(r,t).$$

It should be emphasized that $u_{i,m}(r,t)$ for $m \geqslant 1$ is governed by the linear Eq.(7) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab. Because we have a great freedom to choose $u_{i,0}(r,t)$ in NHAM, we suppose that the initial approximation of the solution of Eq.(1) is in the following form:

$$u_{i,0}(r,t) = f_i(r) + \sum_{j=0}^{+\infty} a_{i,j}(r) \int_{t_0}^{t} P_j(t_1) dt_1, \tag{9}$$

where $a_{i,j}(r)$, are unknown coefficients and $P_0(t),\ P_1(t),\ P_2(t),\ ...$ are specific functions. With substituting Eq.(9) into mth-order deformation Eq.(7) we obtain the other components as follows

(10)

$$\begin{array}{rcl} u_{i,1}(r,t) & = & \hbar_i \ \int_{t_0}^{t} [H_i(r,t_1) R_{i,1} \left(\overrightarrow{u}_{1,0}(r,t_1), \ldots, \overrightarrow{u}_{n,0}(r,t_1)\right)] dt_1, \\ u_{i,2}(r,t) & = & u_{i,1} + \hbar_i \ \int_{t_0}^{t} [H_i(r,t_1) R_{i,2} \left(\overrightarrow{u}_{1,1}(r,t_1), \ldots, \overrightarrow{u}_{n,1}(r,t_1)\right)] dt_1, \\ & \vdots & \\ u_{i,k}(r,t) & = & u_{i,k-1} + \hbar_i \ \int_{t_0}^{t} [H_i(r,t_1) R_{i,k} \left(\overrightarrow{u}_{1,k-1}(r,t_1), \ldots, \overrightarrow{u}_{n,k-1}(r,t_1)\right)] dt_1, \\ & \vdots & \end{array}$$

Now if we solve these equations in such a way that $u_{i,1}(r,t) = 0$, then Eqs.(10) yield

$$u_{i,1}(r,t) = u_{i,2}(r,t) = \cdots = 0.$$

Therefore the exact solution may be obtained as the following

$$u_i(r,t) = u_{i,0}(r,t) = f_i(r) + \sum_{j=0}^{+\infty} a_{i,j}(r) \int_{t_0}^{t} P_j(t) dt_1.$$

It is worth mentioning that if $g_i(x_1, x_2, ..., x_{n-1}, t)$ and $u_{i,0}(x_1, x_2, ..., x_{n-1}, t)$, are analytic around $t = t_0$, then their Taylor series can be defined as

$$u_{i,0}(r,t) = \sum_{j=0}^{+\infty} a_{i,j}(r) \left(t - t_0\right)^n,$$

$$g_i(r,t) = \sum_{j=0}^{+\infty} b_{i,j}(r) \left(t - t_0\right)^n,$$

which can be used in Eqs.(10), where $a_{i,j}(r)$ are unknown coefficients which must be computed, and $b_{i,j}(r)$ are known ones. To show the capability of the method, NHAM has been applied to some examples in the next section. If Eq.(1) admits unique solution, then this method will produce the unique solution and if Eq.(1) does not possess unique solution, the NHAM will give a solution among many other (possible) solutions.

3. Numerical Example

In order to illustrate the effectiveness of the method discussed above, some example of systems of non-linear partial equations are presented.

Example 3.1. Consider the following system of three-dimensional partial differential equations:

$$\left\{\begin{array}{l} \dfrac{\partial y_1}{\partial t} - y_2\dfrac{\partial y_1}{\partial x} - \dfrac{\partial y_1}{\partial y}\dfrac{\partial y_2}{\partial t} = 1 - x + y + t, \\ \\ \dfrac{\partial y_2}{\partial t} - y_1\dfrac{\partial y_2}{\partial x} - \dfrac{\partial y_2}{\partial y}\dfrac{\partial y_1}{\partial t} = 1 - x - y - t, \end{array}\right. \tag{11}$$

the initial conditions are given by

$$y_1(x, y, 0) = x + y - 1, \qquad y_2(x, y, 0) = x - y + 1,$$

the exact solutions are

$$y_1(x, y, t) = x + y + t - 1, \qquad y_2(x, y, t) = x - y - t + 1.$$

First we apply the HAM approach and then the NHAM approach.

HAM approach:

Solving the system (11) by the HAM with $\hbar_i = -1$, $\mathcal{H}_i(x, y, t) = 1$, for $i = 1, 2$, and $y_{1,0}(x, y, t) = x + y - 1$ and $y_{2,0}(x, y, t) = x - y + 1$, we obtain

$$\left\{\begin{array}{l} y_{1,1}(x, y, t) = 2t + \frac{1}{2}t^2, \quad y_{2,1}(x, y, t) = -\frac{1}{2}t^2, \\ \\ y_{2,1}(x, y, t) = -\frac{1}{2}t^2, \end{array}\right.$$

$$\left\{\begin{array}{l} y_{1,2}(x, y, t) = -\frac{1}{6}t^3 - \frac{1}{2}t^2, \qquad y_{2,2}(x, y, t) = \frac{1}{2}t^2 + \frac{1}{6}t^3 - 2t, \\ \\ y_{2,2}(x, y, t) = \frac{1}{2}t^2 + \frac{1}{6}t^3 - 2t, \end{array}\right.$$

$$\left\{\begin{array}{l} y_{1,3}(x, y, t) = \frac{1}{3}t^3 + \frac{1}{24}t^4 - \frac{1}{2}t^2 - 2t, \quad y_{2,3}(x, y, t) = -\frac{1}{24}t^4 + \frac{1}{2}t^2, \\ \\ y_{2,3}(x, y, t) = -\frac{1}{24}t^4 + \frac{1}{2}t^2, \end{array}\right.$$

$$\vdots$$

Therefore, the approximate solution of Example (3.1), can be readily obtained by

$$\begin{cases} y_1(x,y,t)=\sum_{m=0}^{+\infty} y_{1,m}(x,y,t)=-1+x+y+2t-\frac{1}{5040}t^7+\frac{1}{40}t^5-\frac{1}{2}t^3+\cdots, \\ y_2(x,y,t)=\sum_{m=0}^{+\infty} y_{2,m}(x,y,t)=1+x-y+2t+\frac{1}{5040}t^7-\frac{1}{40}t^5+\frac{1}{2}t^3+\cdots. \end{cases}$$

NHAM approach:

To solve the system (11) by the NHAM, for simplicity we define the system of nonlinear operators as

$$\begin{cases} \mathcal{N}_1[\phi_1(x,y,t;p),\phi_2(x,y,t;p)]=\frac{\partial\phi_1}{\partial t}-\phi_2\frac{\partial\phi_1}{\partial x}-\frac{\partial\phi_1}{\partial y}\frac{\partial\phi_2}{\partial t}-1+x-y-t, \\ \mathcal{N}_2[\phi_1(x,y,t;p),\phi_2(x,y,t;p)]=\frac{\partial\phi_2}{\partial t}-\phi_1\frac{\partial\phi_2}{\partial x}-\frac{\partial\phi_2}{\partial y}\frac{\partial\phi_1}{\partial t}-1+x+y+t. \end{cases}$$

Using above definition, we construct the system zeroth-order deformation equations

$$(1-p)\left[\frac{\partial\phi_i(x,y,t;p)}{\partial t}-y_{i,0}(x,y,t)\right]=p\,\hbar_i\,\mathcal{H}_i(x,y,t)\,\mathcal{N}_i[\phi_1,\phi_2],\quad (i=1,2).$$

For $p=0$ and $p=1$, we can write

$$\phi_i(x,y,t;0)=\int_0^t y_{i,0}(x,y,t_1)dt_1,\qquad \phi_i(x,y,t;1)=y_i(x,y,t),\qquad (i=1,2).$$

Thus, we obtain the system of mth-order deformation equations as follows (12)

$$\frac{\partial}{\partial t}[y_{i,m}(x,y,t)-\chi_m y_{i,m-1}(x,y,t)]=\hbar_i\;\mathcal{H}_i(x,y,t)\;\mathcal{R}_{i,m}(\overrightarrow{y}_{1,m-1},\overrightarrow{y}_{2,m-1}),$$

where

$$\begin{cases} \mathcal{R}_{1,m}=\frac{\partial y_{1,m-1}}{\partial t}-\sum_{k=0}^{m-1}\left[y_{2,k}\frac{\partial y_{1,m-1-k}}{\partial x}+\frac{\partial y_{1,k}}{\partial y}\frac{\partial y_{2,m-1-k}}{\partial t}\right] \\ \qquad +(1-\chi_m)(-1+x-y-t), \\ \mathcal{R}_{2,m}=\frac{\partial y_{2,m-1}}{\partial t}-\sum_{k=0}^{m-1}\left[y_{1,k}\frac{\partial y_{2,m-1-k}}{\partial x}+\frac{\partial y_{2,k}}{\partial y}\frac{\partial y_{1,m-1-k}}{\partial t}\right] \\ \qquad +(1-\chi_m)(-1+x+y+t). \end{cases}$$

Now, the solution of system (12), for $(m\geq 1)$ is

$$y_{i,m}(x,y,t)=\chi_m y_{i,m-1}(x,y,t)+\hbar_i\int_0^t[\mathcal{H}_i(x,y,t_1)\mathcal{R}_{i,m}(\overrightarrow{y}_{1,m-1},\overrightarrow{y}_{2,m-1})]\,dt_1,$$

in other words

$$
\begin{cases}
y_{1,1}(x,y,t) = \hbar_1 \int_0^t [\mathcal{H}_1(x,y,t_1)\mathcal{R}_{1,1}(\overrightarrow{y}_{1,0},\overrightarrow{y}_{2,0})]\,dt_1,\\
y_{2,1}(x,y,t) = \hbar_2 \int_0^t [\mathcal{H}_2(x,y,t_1)\mathcal{R}_{2,1}(\overrightarrow{y}_{1,0},\overrightarrow{y}_{2,0})]\,dt_1,
\end{cases}
$$

$$
\begin{cases}
y_{1,2}(x,y,t) = y_{1,1}(x,y,t) + \hbar_1 \int_0^t [\mathcal{H}_1(x,y,t_1)\mathcal{R}_{1,2}(\overrightarrow{y}_{1,1},\overrightarrow{y}_{2,1})]\,dt_1,\\
y_{2,2}(x,y,t) = y_{2,1}(x,y,t) + \hbar_2 \int_0^t [\mathcal{H}_2(x,y,t_1)\mathcal{R}_{2,2}(\overrightarrow{y}_{1,1},\overrightarrow{y}_{2,1})]\,dt_1,
\end{cases}
$$

$$\vdots$$

$$
\begin{cases}
y_{1,j+1}(x,y,t) = y_{1,j}(x,y,t) + \hbar_1 \int_0^t [\mathcal{H}_1(x,y,t_1)\mathcal{R}_{1,j+1}(\overrightarrow{y}_{1,j},\overrightarrow{y}_{2,j})]\,dt_1,\\
y_{2,j+1}(x,y,t) = y_{2,j}(x,y,t) + \hbar_2 \int_0^t [\mathcal{H}_2(x,y,t_1)\mathcal{R}_{2,j+1}(\overrightarrow{y}_{1,j},\overrightarrow{y}_{2,j})]\,dt_1,
\end{cases}
$$

$$\vdots$$

For solving the above systems by means of NHAM, considering that $\mathcal{H}_1(x,y,t_1) = \mathcal{H}_2(x,y,t_1) = 1,\ \hbar_1 = \hbar_2 = -1$, with initial approximations as follows

$$
\begin{cases}
y_{1,0}(x,y,t) = x + y - 1 + \int_0^t \left[\sum_{j=0}^{+\infty} a_j(x,y)P_{1,j}(t)\right],\\
y_{2,0}(x,y,t) = x - y + 1 + \int_0^t \left[\sum_{j=0}^{+\infty} b_j(x,y)P_{2,j}(t)\right],
\end{cases}
$$

such that $P_{i,j}(t) = t^j$, for $i = 1,2$. Solving the above equations for $y_{1,1}(x,y,t)$ and $y_{2,1}(x,y,t)$ with considering the above information leads to the result

$$
\begin{aligned}
y_{1,1}(x,y,t) = {} & [-a_0(x,y) + b_0(x,y) + 2]\,t + [-\frac{1}{2}a_1(x,y) + \frac{1}{2}b_0(x,y)\\
& + \frac{1}{2}(x-y+1)a_{0_x}(x,y) + \frac{1}{2}b_1(x,y) + \frac{1}{2}a_{0_y}(x,y)b_0(x,y) + \frac{1}{2}]t^2\\
& + [-\frac{1}{3}a_2(x,y) + \frac{1}{6}b_1(x,y) + \frac{1}{6}(x-y+1)a_{1_x}(x,y) + \frac{1}{3}a_{0_x}(x,y)\\
& \times b_0(x,y) + \frac{1}{3}b_2(x,y) + \frac{1}{3}a_{0_y}(x,y)b_1(x,y) + \frac{1}{6}a_{1_y}(x,y)b_0(x,y)]t^3\\
& + [-\frac{1}{4}a_3(x,y) + \frac{1}{12}b_2(x,y) + \frac{1}{12}(x-y+1)a_{2_x}(x,y) + \frac{1}{8}a_{0_x}(x,y)\\
& \times b_1(x,y) + \frac{1}{8}a_{1_x}(x,y)b_0(x,y) + \frac{1}{4}b_3(x,y) + \frac{1}{4}a_{0_y}(x,y)b_2(x,y)\\
& + \frac{1}{8}a_{1_y}(x,y)b_1(x,y) + \frac{1}{12}a_{2_y}(x,y)b_0(x,y)]t^4 + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
y_{2,1}(x,y,t) &= (-b_0(x,y) - a_0(x,y))t + [-\frac{1}{2}b_1(x,y) + \frac{1}{2}a_0(x,y) \\
&+ \frac{1}{2}(x+y-1)b_{0_x}(x,y) - \frac{1}{2}a_1(x,y) + \frac{1}{2}a_0(x,y)b_{0_y}(x,y) - \frac{1}{2}]t^2 \\
&+ [-\frac{1}{3}b_2(x,y) + \frac{1}{6}a_1(x,y) + \frac{1}{6}(x+y-1)b_{1_x}(x,y) + \frac{1}{3}a_0(x,y) \\
&\times b_{0_x}(x,y) - \frac{1}{3}a_2(x,y) + \frac{1}{6}a_0(x,y)b_{1_y}(x,y) + \frac{1}{3}a_1(x,y)b_{0_y}(x,y)]t^3 \\
&+ [-\frac{1}{4}b_3(x,y) + \frac{1}{12}a_2(x,y) + \frac{1}{12}(x+y-1)b_{2_x}(x,y) + \frac{1}{8}a_1(x,y) \\
&\times b_{0_x}(x,y) + \frac{1}{8}a_0(x,y)b_{1_x}(x,y) - \frac{1}{4}a_2(x,y) + \frac{1}{4}a_2(x,y)b_{0_y}(x,y) \\
&+ \frac{1}{8}a_1(x,y)b_{1_y}(x,y) + \frac{1}{12}a_0(x,y)b_{2_y}(x,y)]t^4 + \cdots .
\end{aligned}
$$

By the vanishing of $y_{1,1}(x,y,t)$ and $y_{2,1}(x,y,t)$ the coefficients $a_j(x,y)$ and $b_j(x,y)$ for $j = 1,2,3,...$ are determined as

$$
\begin{aligned}
&a_0(x,y) = 1, \qquad a_1(x,y) = a_2(x,y) = a_3(x,y) = \cdots = 0, \\
&b_0(x,y) = -1, \qquad b_1(x,y) = b_2(x,y) = b_3(x,y) = \cdots = 0,
\end{aligned}
$$

therefore we obtain the solution of Eq.(11) as

$$
\begin{aligned}
y_1(x,y,t) &= y_{1,0}(x,y,t) = x + y - 1 + a_0(x,y)t + \frac{1}{2}a_1(x,y)t^2 + \frac{1}{3}a_2(x,y)t^3 \\
&+ \frac{1}{4}a_3(x,y)t^4 + \cdots = x + y + t - 1, \\
y_2(x,y,t) &= y_{2,0}(x,y,t) = x - y + 1 + b_0(x,y)t + \frac{1}{2}b_1(x,y)t^2 + \frac{1}{3}b_2(x,y)t^3 \\
&+ \frac{1}{4}b_3(x,y)t^4 + \cdots = x - y - t + 1,
\end{aligned}
$$

which is an exact solution for Eq.(11).

Example 3.2. Consider the following system of two-dimensional partial differential equations:

$$
\begin{cases}
\dfrac{\partial y_1}{\partial x} - y_2\dfrac{\partial y_1}{\partial t} + y_1\dfrac{\partial y_2}{\partial t} = -1 + e^x \sin t, \\[2ex]
\dfrac{\partial y_2}{\partial x} + \dfrac{\partial y_1}{\partial t}\dfrac{\partial y_2}{\partial x} + \dfrac{\partial y_2}{\partial t}\dfrac{\partial y_1}{\partial x} = -1 - e^{-x}\cos t,
\end{cases}
$$

the boundary conditions are given by

$$
y_1(0,t) = \sin t, \qquad y_2(0,t) = \cos t,
$$

the exact solutions are

$$
y_1(x,t) = e^x \sin t, \qquad y_2(x,t) = e^{-x}\cos t.
$$

First we apply the HAM approach and then the NHAM approach.

HAM approach:

Solving the system (13) by the HAM with $\hbar_i = -1$, $\mathcal{H}_i(x,t) = 1$, for $i = 1,2$, and $y_{1,0}(x,t) = \sin t$ and $y_{2,0}(x,t) = \cos t$, we obtain

$$\begin{cases} y_{1,1}(x,t) = e^x \sin t - \sin t, \\ y_{2,1}(x,t) = -x + e^{-x}\cos t - \cos t, \end{cases}$$
$$\begin{cases} y_{1,2}(x,t) = e^x - e^{-x} - 2x - \frac{1}{2}x^2 \cos t, \\ y_{2,2}(x,t) = -e^{-x}\cos^2 t + e^x \sin^2 t + 2\cos^2 t + x\cos t - 1, \end{cases}$$
$$\vdots$$

Therefore, the approximate solution of Example (3.2), can be readily obtained by

$$\begin{cases} y_1(x,t) = \sum_{m=0}^{+\infty} y_{1,m}(x,t) = e^x \sin t + e^x - e^{-x} - 2x - \frac{1}{2}x^2\cos t + \cdots, \\ y_2(x,t) = \sum_{m=0}^{+\infty} y_{2,m}(x,t) = -e^{-x}\cos^2 t + e^x \sin^2 t + e^{-x}\cos t + 2\cos^2 t \\ \qquad + x\cos t - x - 1 + \cdots. \end{cases}$$

NHAM approach:

To solve the system (13) by the NHAM, for simplicity we define the system of nonlinear operators as

$$\begin{cases} \mathcal{N}_1[\phi_1(x,t;p), \phi_2(x,t;p)] = \frac{\partial \phi_1}{\partial x} - \phi_2 \frac{\partial \phi_1}{\partial t} + \phi_1 \frac{\partial \phi_2}{\partial t} + 1 - e^x \sin t, \\ \mathcal{N}_2[\phi_1(x,t;p), \phi_2(x,t;p)] = \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_1}{\partial t}\frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_2}{\partial t}\frac{\partial \phi_1}{\partial x} + 1 + e^{-x}\cos t. \end{cases}$$

Thus, we obtain the system of mth-order deformation equations as follows

$$(14) \qquad \frac{\partial}{\partial x}[y_{i,m}(x,t) - \chi_m y_{i,m-1}(x,t)] = \hbar_i \; \mathcal{H}_i(x,t) \; \mathcal{R}_{i,m}(\overrightarrow{y}_{1,m-1}, \overrightarrow{y}_{2,m-1}),$$

where

$$\begin{cases} \mathcal{R}_{1,m}(\overrightarrow{y}_{1,m-1}, \overrightarrow{y}_{2,m-1}) = \frac{\partial y_{1,m-1}}{\partial x} + \sum_{k=0}^{m-1}\left[y_{1,k}\frac{\partial y_{2,m-1-k}}{\partial t} - y_{2,k}\frac{\partial y_{1,m-1-k}}{\partial t}\right] \\ \qquad + (1-\chi_m)(1 - e^x \sin t), \\ \mathcal{R}_{2,m}(\overrightarrow{y}_{1,m-1}, \overrightarrow{y}_{2,m-1}) = \frac{\partial y_{2,m-1}}{\partial x} + \sum_{k=0}^{m-1}\left[\frac{\partial y_{1,k}}{\partial t}\frac{\partial y_{2,m-1-k}}{\partial x} + \frac{\partial y_{2,k}}{\partial t}\frac{\partial y_{1,m-1-k}}{\partial x}\right] \\ \qquad + (1-\chi_m)(1 + e^{-x}\cos t). \end{cases}$$

Now, the solution of system (14), for $(m \geq 1)$ is

$$y_{i,m}(x,t) = \chi_m y_{i,m-1}(x,t) + \ \hbar_i \int_0^x \left[\mathcal{H}_i(x_1,t)\mathcal{R}_{i,m}(\overrightarrow{y}_{1,m-1}, \overrightarrow{y}_{2,m-1})\right] dx_1,$$

in other words

$$\begin{cases} y_{1,1}(x,t) = \hbar_1 \ \int_0^x \left[\mathcal{H}_1(x_1,t)\mathcal{R}_{1,1}(\overrightarrow{y}_{1,0}, \overrightarrow{y}_{2,0})\right] dx_1, \\ y_{2,1}(x,t) = \hbar_2 \ \int_0^x \left[\mathcal{H}_2(x_1,t)\mathcal{R}_{2,1}(\overrightarrow{y}_{1,0}, \overrightarrow{y}_{2,0})\right] dx_1, \end{cases}$$

$$\begin{cases} y_{1,2}(x,t) = y_{1,1}(x,t) + \hbar_1 \ \int_0^x \left[\mathcal{H}_1(x_1,t)\mathcal{R}_{1,2}(\overrightarrow{y}_{1,1}, \overrightarrow{y}_{2,1})\right] dx_1, \\ y_{2,2}(x,t) = y_{2,1}(x,t) + \hbar_2 \ \int_0^x \left[\mathcal{H}_2(x_1,t)\mathcal{R}_{2,2}(\overrightarrow{y}_{1,1}, \overrightarrow{y}_{2,1})\right] dx_1, \end{cases}$$

$$\vdots$$

$$\begin{cases} y_{1,j+1}(x,t) = y_{1,j}(x,t) + \hbar_1 \ \int_0^x \left[\mathcal{H}_1(x_1,t)\mathcal{R}_{1,j+1}(\overrightarrow{y}_{1,j}, \overrightarrow{y}_{2,j})\right] dx_1, \\ y_{2,j+1}(x,t) = y_{2,j}(x,t) + \hbar_2 \ \int_0^x \left[\mathcal{H}_2(x_1,t)\mathcal{R}_{2,j+1}(\overrightarrow{y}_{1,j}, \overrightarrow{y}_{2,j})\right] dx_1, \end{cases}$$

$$\vdots$$

For solving the above systems by means of NHAM, considering that $\mathcal{H}_1(x_1,t) = \mathcal{H}_2(x_1,t) = 1, \ \hbar_1 = \hbar_2 = -1$, with initial approximations as follows

$$\begin{cases} y_{1,0}(x,t) = \sin t + \int_0^x \left[\sum_{j=0}^{+\infty} a_j(t) P_{1,j}(x_1)\right], \\ y_{2,0}(x,t) = \cos t + \int_0^x \left[\sum_{j=0}^{+\infty} b_j(t) P_{2,j}(x_1)\right], \end{cases}$$

such that $P_{i,j}(x) = x^j$, for $i = 1,2$. Solving the above equations for $y_{1,1}(x,t)$ and $y_{2,1}(x,t)$ with considering the above information leads to the result

$$\begin{aligned} y_{1,1}(x,t) = {} & [-a_0(t) + \sin t]\, x + [-\frac{1}{2}a_1(t) + \frac{a_0^{'}(t)\cos t}{2} + \frac{b_0(t)\cos t}{2} - \frac{b_0^{'}(t)\sin t}{2} \\ & + \frac{a_0(t)\sin t}{2} + \frac{\sin t}{2}]x^2 + [-\frac{1}{3}a_2(t) + \frac{a_1^{'}(t)\cos t}{6} + \frac{b_1(t)\cos t}{6} \\ & + \frac{b_0(t)a_0^{'}(t)}{3} - \frac{b_1^{'}(t)\sin t}{6} + \frac{a_1(t)\sin t}{6} - \frac{b_0^{'}(t)a_0(t)}{3} + \frac{\sin t}{6}]x^3 \\ & + [-\frac{1}{4}a_3(t) + \frac{a_2^{'}(t)\cos t}{12} + \frac{b_2(t)\cos t}{12} + \frac{b_1(t)a_0^{'}(t)}{8} + \frac{b_0(t)a_1^{'}(t)}{8} \\ & - \frac{b_2^{'}(t)\sin t}{12} + \frac{a_2(t)\sin t}{12} - \frac{b_0^{'}(t)a_1(t)}{8} - \frac{b_1^{'}(t)a_0(t)}{8} + \frac{\sin t}{24}]x^4 + \cdots, \end{aligned}$$

$$\begin{aligned} y_{2,1}(x,t) = & [-b_0(t) - b_0(t)\cos t + a_0(t)\sin t - \cos t - 1]x + [-\frac{1}{2}b_1(t) \\ & - \frac{b_1(t)\cos t}{2} - \frac{b_0(t)a_0^{'}(t)}{2} + \frac{a_1(t)\sin t}{2} - \frac{b_0^{'}(t)a_0(t)}{2} + \frac{\cos t}{2}]x^2 \\ & + [-\frac{1}{3}b_2(t) - \frac{b_2(t)\cos t}{3} - \frac{b_0(t)a_1^{'}(t)}{6} - \frac{b_1(t)a_0^{'}(t)}{3} + \frac{a_2(t)\sin t}{3} \\ & - \frac{b_1^{'}(t)a_0(t)}{6} - \frac{b_0^{'}(t)a_1(t)}{3} - \frac{\cos t}{6}]x^3 + [-\frac{1}{4}b_3(t) - \frac{b_3(t)\cos t}{4} \\ & - \frac{b_0(t)a_2^{'}(t)}{12} - \frac{b_1(t)a_1^{'}(t)}{8} - \frac{b_2(t)a_0^{'}(t)}{4} + \frac{a_3(t)\sin t}{4} - \frac{b_0^{'}(t)a_2(t)}{4} \\ & - \frac{b_1^{'}(t)a_1(t)}{8} - \frac{b_2^{'}(t)a_0(t)}{12} + \frac{\cos t}{24}]x^4 + \cdots . \end{aligned}$$

By the vanishing of $y_{1,1}(x,t)$ and $y_{2,1}(x,t)$ the coefficients $a_j(t)$ and $b_j(t)$ for $j = 1, 2, 3, ...$ are determined as

$$\begin{cases} a_0(t) = \sin t, & a_1(t) = \sin t, & a_2(t) = \frac{1}{2}\sin t, \\ a_3(t) = \frac{1}{6}\sin t, & a_4(t) = \frac{1}{24}\sin t, & \cdots \end{cases}$$

$$\begin{cases} b_0(t) = -\cos t, & b_1(t) = \cos t, & b_2(t) = -\frac{1}{2}\cos t, \\ b_3(t) = \frac{1}{6}\cos t, & b_4(t) = -\frac{1}{24}\cos t, & \cdots \end{cases}$$

Therefore we obtain the solution of Eq.(13) as

$$\begin{aligned} y_1(x,t) &= y_{1,0}(x,t) = \sin t + a_0(t)x + \frac{1}{2}a_1(t)x^2 + \frac{1}{3}a_2(t)x^3 + \cdots = e^x \sin t, \\ y_2(x,t) &= y_{2,0}(x,t) = \cos t + b_0(t)x + \frac{1}{2}b_1(t)x^2 + \frac{1}{3}b_2(t)x^3 + \cdots = e^{-x}\cos t, \end{aligned}$$

which is an exact solution for Eq.(13).

Example 3.3. Consider the following non-linear system of inhomogeneous partial differential equations:

$$(15) \qquad \begin{cases} \frac{\partial y_1}{\partial t} - \frac{\partial y_3}{\partial x}\frac{\partial y_2}{\partial t} - \frac{1}{2}\frac{\partial y_3}{\partial t}\frac{\partial^2 y_1}{\partial x^2} = -4xt, \\ \frac{\partial y_2}{\partial t} - \frac{\partial y_3}{\partial t}\frac{\partial^2 y_1}{\partial x^2} = 6t, \\ \frac{\partial y_3}{\partial t} - \frac{\partial^2 y_1}{\partial x^2} - \frac{\partial y_2}{\partial x}\frac{\partial y_3}{\partial t} = 4xt - 2t - 2, \end{cases}$$

the initial conditions are given by

$$y_1(x,0) = x^2 + 1, \qquad y_2(x,0)x^2 - 1, \qquad y_3(x,0) = x^2 - 1,$$

the exact solutions are

$$y_1(x,t) = x^2 - t^2 + 1, \qquad y_2(x,t) = x^2 + t^2 - 1, \qquad y_3(x,t) = x^2 - t^2 - 1.$$

First we apply the HAM approach and then the NHAM approach.

HAM approach:

Solving the system (15) by the HAM with $\hbar_i = -1$, $\mathcal{H}_i(x,t) = 1$, for $i = 1,2,3$, and $y_{1,0}(x,t) = x^2+1$, $y_{2,0}(x,t) = x^2-1$, and $y_{3,0}(x,t) = x^2-1$, we obtain

$$\begin{cases} y_{1,1}(x,t) = -2xt^2, \\ y_{2,1}(x,t) = 3t^2, \\ y_{3,1}(x,t) = 2xt^2 - t^2, \end{cases}$$
$$\begin{cases} y_{1,2}(x,t) = \frac{1}{2}[16x-2]t^2, \\ y_{2,2}(x,t) = \frac{1}{2}[8x-4]t^2, \\ y_{3,2}(x,t) = x[4x-2]t^2, \end{cases}$$
$$\begin{cases} y_{1,3}(x,t) = 3t^4 + [x(8x-4) + x(4x-2)]t^2, \\ y_{2,3}(x,t) = 2x[4x-2]t^2, \\ y_{3,2}(x,t) = 2x^2[4x-2]t^2, \end{cases}$$
$$\vdots$$

Therefore, the approximate solution of Example (3.3), can be readily obtained by

$$\begin{cases} y_1(x,t) = \sum_{m=0}^{+\infty} y_{1,m}(x,t) \approx x^2 - t^2 + 1, \\ y_2(x,t) = \sum_{m=0}^{+\infty} y_{2,m}(x,t) \approx x^2 + t^2 - 1, \\ y_3(x,t) = \sum_{m=0}^{+\infty} y_{3,m}(x,t) \approx x^2 - t^2 - 1. \end{cases}$$

NHAM approach:

To solve the system (15) by the NHAM, for simplicity we define the system of nonlinear operators as

$$\begin{cases} \mathcal{N}_1[\phi_1(x,t;p), \phi_2(x,t;p), \phi_3(x,t;p)] = \frac{\partial \phi_1}{\partial t} - \frac{\partial \phi_3}{\partial x}\frac{\partial \phi_2}{\partial t} - \frac{1}{2}\frac{\partial \phi_3}{\partial t}\frac{\partial^2 \phi_1}{\partial x^2} + 4xt, \\ \mathcal{N}_2[\phi_1(x,t;p), \phi_2(x,t;p), \phi_3(x,t;p)] = \frac{\partial \phi_2}{\partial t} - \frac{\partial \phi_3}{\partial t}\frac{\partial^2 \phi_1}{\partial x^2} - 6t, \\ \mathcal{N}_3[\phi_1(x,t;p), \phi_2(x,t;p), \phi_3(x,t;p)] = \frac{\partial \phi_3}{\partial t} - \frac{\partial^2 \phi_1}{\partial x^2} - \frac{\partial \phi_2}{\partial x}\frac{\partial \phi_3}{\partial t} - 4xt + 2t + 2. \end{cases}$$

Thus, we obtain the system of mth-order deformation equations as follows (16)

$$\frac{\partial}{\partial t}[y_{i,m}(x,t)-\chi_m y_{i,m-1}(x,t)]=\hbar_i\ \mathcal{H}_i(x,t)\ \mathcal{R}_{i,m}(\overrightarrow{y}_{1,m-1},\overrightarrow{y}_{2,m-1},\overrightarrow{y}_{3,m-1}),$$

where

$$\begin{cases} \mathcal{R}_{1,m}(\overrightarrow{y}_{1,m-1},\overrightarrow{y}_{2,m-1},\overrightarrow{y}_{3,m-1}) = \frac{\partial y_{1,m-1}}{\partial t}-\sum_{k=0}^{m-1}[\frac{\partial y_{3,k}}{\partial x}\frac{\partial y_{2,m-1-k}}{\partial t} \\ \qquad\qquad + \frac{1}{2}\frac{\partial y_{3,k}}{\partial t}\frac{\partial^2 y_{1,m-1-k}}{\partial x^2}]+(1-\chi_m)(4xt), \\ \mathcal{R}_{2,m}(\overrightarrow{y}_{1,m-1},\overrightarrow{y}_{2,m-1},\overrightarrow{y}_{3,m-1}) = \frac{\partial y_{2,m-1}}{\partial t}-\sum_{k=0}^{m-1}\left[\frac{\partial y_{3,k}}{\partial t}\frac{\partial^2 y_{1,m-1-k}}{\partial x^2}\right] \\ \qquad\qquad +(1-\chi_m)(-6t), \\ \mathcal{R}_{3,m}(\overrightarrow{y}_{1,m-1},\overrightarrow{y}_{2,m-1},\overrightarrow{y}_{3,m-1}) = \frac{\partial y_{3,m-1}}{\partial t}-\frac{\partial^2 y_{1,m-1}}{\partial x^2}-\sum_{k=0}^{m-1}\left[\frac{\partial y_{2,k}}{\partial x}\frac{\partial y_{3,m-1-k}}{\partial t}\right] \\ \qquad\qquad +(1-\chi_m)(-4xt+2t+2). \end{cases}$$

Now, the solution of system (16), for $(m\geq 1)$ is

$$y_{i,m}(x,t)=\chi_m y_{i,m-1}(x,t)+\hbar_i\int_0^t[\mathcal{H}_i(x,t_1)\mathcal{R}_{i,m}(\overrightarrow{y}_{1,m-1},\overrightarrow{y}_{2,m-1},\overrightarrow{y}_{3,m-1})]\,dt_1,$$

in other words

$$\begin{cases} y_{1,1}(x,t)=\hbar_1\ \int_0^t[\mathcal{H}_1(x,t_1)\mathcal{R}_{1,1}(\overrightarrow{y}_{1,0},\overrightarrow{y}_{2,0},\overrightarrow{y}_{3,0})]\,dt_1, \\ y_{2,1}(x,t)=\hbar_2\ \int_0^t[\mathcal{H}_2(x,t_1)\mathcal{R}_{2,1}(\overrightarrow{y}_{1,0},\overrightarrow{y}_{2,0},\overrightarrow{y}_{3,0})]\,dt_1, \\ y_{3,1}(x,t)=\hbar_3\ \int_0^t[\mathcal{H}_3(x,t_1)\mathcal{R}_{3,1}(\overrightarrow{y}_{1,0},\overrightarrow{y}_{2,0},\overrightarrow{y}_{3,0})]\,dt_1, \end{cases}$$

$$\begin{cases} y_{1,2}(x,t)=y_{1,1}(x,t)+\hbar_1\ \int_0^t[\mathcal{H}_1(x,t_1)\mathcal{R}_{1,2}(\overrightarrow{y}_{1,1},\overrightarrow{y}_{2,1},\overrightarrow{y}_{3,1})]\,dt_1, \\ y_{2,2}(x,t)=y_{2,1}(x,t)+\hbar_2\ \int_0^t[\mathcal{H}_2(x,t_1)\mathcal{R}_{2,2}(\overrightarrow{y}_{1,1},\overrightarrow{y}_{2,1},\overrightarrow{y}_{3,1})]\,dt_1, \\ y_{3,2}(x,t)=y_{3,1}(x,t)+\hbar_3\ \int_0^t[\mathcal{H}_3(x,t_1)\mathcal{R}_{3,2}(\overrightarrow{y}_{1,1},\overrightarrow{y}_{2,1},\overrightarrow{y}_{3,1})]\,dt_1, \end{cases}$$

$$\vdots$$

$$\begin{cases} y_{1,j+1}(x,t)=y_{1,j}(x,t)+\hbar_1\ \int_0^t[\mathcal{H}_1(x,t_1)\mathcal{R}_{1,j+1}(\overrightarrow{y}_{1,j},\overrightarrow{y}_{2,j},\overrightarrow{y}_{3,j})]\,dt_1, \\ y_{2,j+1}(x,t)=y_{2,j}(x,t)+\hbar_2\ \int_0^t[\mathcal{H}_2(x,t_1)\mathcal{R}_{2,j+1}(\overrightarrow{y}_{1,j},\overrightarrow{y}_{2,j},\overrightarrow{y}_{3,j})]\,dt_1, \\ y_{3,j+1}(x,t)=y_{3,j}(x,t)+\hbar_3\ \int_0^t[\mathcal{H}_3(x,t_1)\mathcal{R}_{3,j+1}(\overrightarrow{y}_{1,j},\overrightarrow{y}_{2,j},\overrightarrow{y}_{3,j})]\,dt_1, \end{cases}$$

$$\vdots$$

For solving the above systems by means of NHAM, considering that $\mathcal{H}_1(x,t_1) = \mathcal{H}_2(x,t_1) = \mathcal{H}_3(x,t_1) = 1,\ \hbar_1 = \hbar_2 = \hbar_3 = -1$, with initial approximations as follows

$$\begin{cases} y_{1,0}(x,t) = x^2 + 1 + \int_0^t \left[\sum_{j=0}^{+\infty} a_j(x)P_{1,j}(t_1)\right], \\ y_{2,0}(x,t) = x^2 - 1 + \int_0^t \left[\sum_{j=0}^{+\infty} b_j(x)P_{2,j}(t_1)\right], \\ y_{3,0}(x,t) = x^2 - 1 + \int_0^t \left[\sum_{j=0}^{+\infty} c_j(x)P_{3,j}(t_1)\right], \end{cases}$$

such that $P_{i,j}(x) = t^j$, for $i = 1, 2, 3$. Solving the above equations for $y_{1,1}(x,t),\ y_{2,1}(x,t)$ and $y_{3,1}(x,t)$ with considering the above information leads to the result

$$\begin{aligned} y_{1,1}(x,t) = {} & [-a_0(x) + 2xb_0(x) + c_0(x)]t + [-\frac{1}{2}a_1(x) + xb_1(x) + \frac{1}{2}b_0(x)c_0^{'}(x) \\ & + \frac{1}{2}c_1(x) + \frac{1}{4}a_0^{''}(x)c_0(x) - 2x]t^2 + [-\frac{1}{3}a_2(x) + \frac{2}{3}xb_2(x) \\ & + \frac{1}{3}b_1(x)c_0^{'}(x) + \frac{1}{6}b_0(x)c_1^{'}(x) + \frac{1}{3}c_2(x) + \frac{1}{12}a_1^{''}(x)c_0(x) \\ & + \frac{1}{6}a_0^{''}(x)c_1(x)]t^3 + [-\frac{1}{4}a_3(x) + \frac{1}{2}xb_3(x) + \frac{1}{4}b_2(x)c_0^{'}(x) \\ & + \frac{1}{8}b_1(x)c_1^{'}(x) + \frac{1}{12}b_0(x)c_2^{'}(x) + \frac{1}{4}c_3(x) + \frac{1}{24}a_2^{''}(x)c_0(x) \\ & + \frac{1}{16}a_1^{''}(x)c_1(x) + \frac{1}{8}a_0^{''}(x)c_2(x)]t^4 + \cdots, \end{aligned}$$

$$\begin{aligned} y_{2,1}(x,t) = {} & [-b_0(x) + 2c_0(x)]t + [-\frac{1}{2}b_1(x) + c_1(x) + \frac{1}{2}c_0(x)a_0^{''}(x) + 3]t^2 \\ & + [-\frac{1}{3}b_2(x) + \frac{2}{3}c_2(x) + \frac{1}{6}c_0(x)a_1^{''}(x) + \frac{1}{3}c_1(x)a_0^{''}(x)]t^3 + [-\frac{1}{4}b_3(x) \\ & + \frac{1}{2}c_3(x) + \frac{1}{12}c_0(x)a_2^{''}(x) + \frac{1}{8}c_1(x)a_1^{''}(x) + \frac{1}{4}c_2(x)a_0^{''}(x)]t^4 + \cdots, \end{aligned}$$

$$\begin{aligned} y_{3,1}(x,t) = {} & [-c_0(x) + 2xc_0(x)]t + [-\frac{1}{2}c_1(x) + \frac{1}{2}a_0^{''}(x) + xc_1(x) + b_0^{'}(x)c_0(x) \\ & + 2x - 1]t^2 + [-\frac{1}{3}c_3(x) + \frac{1}{6}a_1^{''}(x) + \frac{2}{3}xc_2(x) + \frac{1}{3}b_0^{'}(x)c_1(x) \\ & + \frac{1}{6}b_1^{'}(x)c_0(x)]t^3 + [-\frac{1}{4}c_3(x) + \frac{1}{12}a_2^{''}(x) + \frac{1}{2}xc_3(x) + \frac{1}{4}b_0^{'}(x)c_2(x) \\ & + \frac{1}{8}b_1^{'}(x)c_1(x) + \frac{1}{12}b_2^{'}(x)c_0(x)]t^4 + \cdots. \end{aligned}$$

By the vanishing of $y_{1,1}(x,t), y_{2,1}(x,t)$ and $y_{3,1}(x,t)$ the coefficients $a_j(t),\ b_j(t)$

and $c_j(t)$ for $j = 1, 2, 3, ...$ are determined as

$$\begin{cases} a_0(x) = 0, & a_1(x) = -2, & a_2(x) = a_3(x) = a_4(x) = \cdots = 0, \\ b_0(x) = 0, & b_1(x) = 2, & b_2(x) = b_3(x) = b_4(x) = \cdots = 0, \\ c_0(x) = 0, & c_1(x) = -2, & c_2(x) = c_3(x) = c_4(x) = \cdots = 0, \end{cases}$$

Therefore we obtain the solution of Eq.(15) as

$$\begin{aligned} y_1(x,t) &= y_{1,0}(x,t) = x^2 + 1 + a_0(x)t + \frac{1}{2}a_1(x)t^2 + \frac{1}{3}a_2(x)t^3 + \cdots = x^2 - t^2 + 1, \\ y_2(x,t) &= y_{2,0}(x,t) = x^2 - 1 + b_0(x)t + \frac{1}{2}b_1(x)t^2 + \frac{1}{3}b_2(x)t^3 + \cdots = x^2 + t^2 - 1, \\ y_3(x,t) &= y_{3,0}(x,t) = x^2 - 1 + c_0(x)t + \frac{1}{2}c_1(x)t^2 + \frac{1}{3}c_2(x)t^3 + \cdots = x^2 - t^2 - 1, \end{aligned}$$

which is an exact solution for Eq.(15).

4. Conclusion

In this article, a new modification of HAM, called NHAM, has been introduced for solving systems of non-linear partial differential equations. This method has been applied to three examples successfully, and exact solutions of the equations are achieved, where traditional HAM leads to an approximate solution. The examples in this paper are further confirmation of the flexibility and potential of the NHAM for solving complicated linear and non-linear initial and boundary value problems in science and engineering. The computations associated with the examples were performed using Maple. This new method easily can be employed to solve other functional equations.

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