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Taylor Series Approach for Bi-Level Linear Fractional Programming Problem

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Abstract. This paper presents a solution of bi-level linear fractional programming problems (BLLFPP) using of Taylor series. The Taylor series is a series expansion that a representation of a function. Levels are classified as upper level and lower level, and they are weighted with respect to their classes before Taylor series approach unified levels by using their weights. Thus, the problem is reduced to a single objective. Numerical example is provided to demonstrate the efficiency and feasibility of the proposed approach.

Key words: Fractional programming; Series expansions; Management decision making; Linear programming.
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1. Introduction

The Fractional Programming (FP) problem, which has been used as an important planning tool for the past four decades, is applied to different disciplines such as engineering, business, finance, economics, etc. FP is generally used for modeling real life problems with one or more objective(s) such as profit/cost, inventory/sales, actual cost/standard cost, output/employee etc [17]. Multiple level programming problems are frequently encountered in any hierarchical organization of large companies such as government agencies, profit or non-profit organizations, manufacturing plants, logistic companies, etc [1]
A bi-level programming problem (BLPP) [7, 13-15] has two levels, which are the first level and the second level. Bi-level decentralized programming problem (BLDPP) [2] is characterized by a center that controls some (more than one) divisions on the second level. These divisions are independent. The first level decision maker (DM) is called the center. The second level DM called follower,

executes its policies after the decision of higher level DM called leader (center) and then the leader optimizes its objective independently but may be affected by the reaction of the follower, i.e., BLPP is a sequence of two optimization problems in which the constraints region of one is determined by the solution of second [13].
In the literature, many researchers [3-9, 13-15, 18] have focused to solve BLPPs. Some of them [3-5, 8] presented formulation and different version of problem. Bialas and Karwan [7] introduced firstly BLPP who presented vertex enumeration method, which called K*th* best solution. These were solved by simplex method. Ben-Ayed and Blair [6] showed that the parametric complementary pivot, which was proposed by Bialas and Karwan [7], may not converge to optimality. When Bard and Falk [5] proposed the grid search algorithm, Unlu [18] proposed a technique of bi-criteria programming based on Bard and Falk's [5] algorithm. Chakraborty and Gupta [9] proposed fuzzy mathematical programming approach for solving multi-objective linear fractional programming problem when Sakawa and Nishizaki [15] proposed a linear programming based on interactive fuzzy programming for bi-level linear fractional programming. This method is used to derive the satisfying solution for the DM efficiently from a Pareto optimal solution set by updating the reference membership value of the DM. Furthermore, Mishra and Ghosh [14] presented interactive fuzzy programming approach to bi-level quadratic fractional programming problems.
In this paper, both objectives are transformed by using 1^{st} order Taylor polynomial series. Here, the Taylor series obtains polynomial objective functions which are equivalent to fractional objective functions. Thus, BLLFPP can be reduced into a single function. In other words, suitable transformations can be applied to formulate bi-level programming. In the compromised function, the first level are weighted more than second level because the first level decision maker (DM) is called the center. The performance of the proposed method was experimentally validated by numerical example considered by Malhotra and Arora [12]. Results demonstrate that the proposed approach runs more effectively.

2. Formulation of the problem

A bi-level programming problem (BLPP) [3, 4, 7, 13-15] has two levels, which are the first level and the second level. Bi-level decentralized programming problem (BLDPP) [2] is characterized by a center that controls some (more than one) divisions on the second level. These divisions are independent. A multi level programming problem (MLPP) [16-11] can be defined as a p-person, non-zero sum game with perfect information in which each player moves sequentially from top to bottom. This problem is a nested hierarchical structure. When p = 2, we call as a bi-level programming problem. The first level decision maker (DM) is called the center. The second level DM called follower, executes its policies after the decision of higher level DM called leader (center) and then the leader optimizes its objective independently but may be affected by the reaction of the follower, i.e., BLPP is a sequence of two optimization problems in which the constraints region of one is determined by the solution of second [13]. Mishra

[13] gave an example of BLPP by adopting a criterion with respect to finance or corporate planning as an objective function at the upper level and employing a criterion regarding production planning as an objective function at the lower level. A bi-level linear fractional programming problem is formulated as follow:

$$\max_{(x_1)} \text{imize} \quad z_1 (x_1, x_2)$$

$$\max_{(x_2)} \text{imize} \quad z_2 (x_1, x_2)$$

$$\text{subject to} \quad \begin{array}{l} A_1 x_1 + A_2 x_2 \leq b \\ x_1, x_2 \geq 0 \end{array} \tag{1}$$

where $z_1 (x_1, x_2)$ and $z_2 (x_1, x_2)$ respectively represent objective functions of DM_1 and DM_2, and $x_1$and x_2 represent decision variables under the control of DM_1 and DM_2 respectively. Furthermore, let DM_1 denote the DM at the upper level when DM_2denote the DM at the lower level.
Objective functions $z_i (x_1, x_2)$, $(i = 1, 2)$, are represented by a linear fractional function as follows:

$$z_i (x_1, x_2) = \frac{p_i (x_1, x_2)}{q_i (x_1, x_2)} \tag{2}$$

where $x_1$and x_2 represent decision variables.

3. Taylor series approach for bi-level linear fractional programming

In the BLLFPP, objective functions are transformed by using Taylor series at first, and then a satisfactory value(s) for the variable(s) of the model is obtained by solving the model, which has a single objective function. Here, Taylor series obtains polynomial objective functions which are equivalent to fractional objective functions. Then, the BLLFPP can be reduced into a single objective. In the compromised objective function, weight of the first level is more than weight of second level because the first level decision maker (DM) is called the center

The proposed approach, which is inspired approach of Guzel and Sivri [10], can be explained in three steps.

Step 1. Determine $x_i^* = (x_{i1}^*, x_{i2}^*, \ldots, x_{in}^*)$ which is the value(s) that is used to maximize the ith objective function $z_i (x) \quad (i = 1, 2, ...k)$ where n is the number of the variable.

Step 2. Transform objective functions by using 1^{st} order Taylor polynomial series.

$$z_i(x) \cong \hat{z}_i(x) = z_i(x_i^*) + \left[(x_1 - x_{i1}^*) \frac{\partial z_i(x_i^*)}{\partial x_1} + (x_2 - x_{i2}^*) \frac{\partial z_i(x_i^*)}{\partial x_2} \right.$$
$$\left. + ... + (x_n - x_{in}^*) \frac{\partial z_i(x_i^*)}{\partial x_n} \right]$$

$$z_i(x) \cong \hat{z}_i(x) = z_i(x_i^*) + \sum_{j=1}^{n} \left(x_j - x_{ij}^* \right) \frac{\partial z_i(x_i^*)}{\partial x_j} \tag{3}$$

Step 3. Find satisfactory $x_* = (x_1^*, x_2^*, ..., x_n^*)$ by solving the reduced problem to a single objective. In the compromised objective function, weight of the first level is more than weight of second level because the first level decision maker (DM) is called the center.

$$P(x) = \sum_{i=1}^{k} z_i(x_i^*) + \sum_{j=1}^{n} \left(x_j - x_{ij}^* \right) \frac{\partial z_i(x_i^*)}{\partial x_j} \tag{4}$$

BLLFPP is converted into a new mathematical model. This model is as follows:

$$\max \sum_{i=1}^{k} z_i(x_i^*) + \sum_{j=1}^{n} \left(x_j - x_{ij}^* \right) \frac{\partial z_i(x_i^*)}{\partial x_j}$$

$$\text{subject to} \quad \begin{array}{l} A_1 x_1 + A_2 x_2 \le b \\ x_1, x_2 \ge 0 \end{array} \tag{5}$$

Thus a new model, which is equal to the BLLFPP, is obtained.

4. Numerical example

The following example studied by Malhotra and Arora [12] and Mishra [13] is considered to demonstrate the effectiveness of the proposed Taylor series approach

$$\max\text{imize} \quad z_1(x_1, x_2) = \frac{x_1 + 2x_2}{x_1 + x_2 + 1}$$

$$\max\text{imize} \quad z_2(x_1, x_2) = \frac{2x_1 + x_2}{2x_1 + 3x_2 + 1}$$

$$\text{Subject to} \quad -x_1 + 2x_2 \le 3,$$

$$2x_1 - x_2 \leq 3,$$

$$x_1 + x_2 \geq 3,$$

$$x_1, x_2 \geq 0. \tag{6}$$

If the problem is solved for each of the membership functions one by one then $z_1^* (3,3)$ and $z_2^* (2,1)$.
Then objective functions are transformed by using 1$^{\text{st}}$ order Taylor polynomial series.

$$z_1 (x) \cong \hat{z}_1 (x) = z_1 (3,3) + \left[(x_1 - 3) \frac{\partial z_1 (3,3)}{\partial x_1} + (x_2 - 3) \frac{\partial z_1 (3,3)}{\partial x_2} \right]$$

$$z_1 (x) \cong \hat{z}_1 (x) = -0.04x_1 + 0.10x_2 + 1.10 \tag{7}$$

$$z_2 (x) \cong \hat{z}_2 (x) = z_2 (2,1) + \left[(x_1 - 2) \frac{\partial z_2 (2,1)}{\partial x_1} + (x_2 - 1) \frac{\partial z_2 (2,1)}{\partial x_2} \right]$$

$$z_2 (x) \cong \hat{z}_2 (x) = 0.09x_1 - 0.11x_2 + 0.56 \tag{8}$$

The total of (7) and (8) is the objective of BLLFPP under w_1, weight of the first level, is equal 0.51 and w_2, weight of the second level, is equal 0.49. Since, in the compromised objective function, weight of the first level must be more than weight of second level due to the first level decision maker (DM) is called the center.

$$Z (x) = (0.51 \times \hat{z}_1 (x)) + (0.49 \times \hat{z}_2 (x)) = 0.024x_1 - 0.003x_2 + 0.84 \tag{9}$$

Thus, the final form of the BLLFPP is obtained as follows:

$$\text{Find} \;\; x (x_1, x_2) \;\; \text{so as to}$$

$$\max\text{imize} \quad Z (x) = (0.51 \times \hat{z}_1 (x)) + (0.49 \times \hat{z}_2 (x)) = 0.024x_1 - 0.003x_2 + 0.84$$

$$\text{Subject to} \quad -x_1 + 2x_2 \leq 3,$$

$$2x_1 - x_2 \leq 3,$$

$$x_1 + x_2 \geq 3,$$

$$x_1, x_2 \geq 0. \tag{10}$$

The problem is solved and the solution of the above problem is as follows:

$$x_1 = 3, \quad x_2 = 3.$$

The approach of Malhotra and Arora [12] started $(x_1, x_2) = (2, 1)$ at initial stage and obtained $(x_1, x_2) = (3, 3)$ at final stage. Mishra [13] obtained the same solution. Mishra [13] was reduced the non-dominated solution set to a point by using weighting method Proposed approach in this paper got exactly the same solution by transforming objective functions by using 1^{st} order Taylor polynomial series. However, the proposed approach facilitates computation to reduce the complexity in problem solving.

5. Conclusion

In this paper, a powerful and robust approach which is based on Taylor series is proposed to solve bi-level linear fractional programming problems (BLLFPP). Objective functions of the problem are transformed using Taylor series. BLLFPP is reduced to an equivalent multi-objective linear programming problem (MOLP) by using the 1^{st} Taylor polynomial series. The obtained MOLP problem is solved when weight of the first level is more than weight of second level in the compromised objective function because the first level decision maker (DM) is called the center. The proposed solution approach was applied to a numerical example to test its performance. The results show that the proposed approach is more effective when compared to the previous approach. The proposed approach facilitates computation to reduce the complexity in problem solving.

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