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Estimating System Reliability of Competing Weibull Failures with Censored Sampling

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Abstract. In this paper, we consider the estimation of $R = P(Y < X)$ where X and Y have two independent Weibull distributions with different scale parameters and the same shape parameter. We used different methods for estimating R. Assuming that the common shape parameter is known, the maximum likelihood, uniformly minimum variance unbiased and Bayes estimators for R are obtained based on type-II right censored sample. Monte Carlo simulations are performed to compare the different estimators.

Key words: Stress-strength model; Maximum likelihood estimator; Unbiasedness; Consistency; Uniformly minimum variance unbiased estimator; Bayesian estimator; Type II censoring.
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Acronyms and Abbreviations

CDF	Cumulative distribution function
PDF	Probability density function
MLE	Maximum likelihood estimator
UMVUE	Uniformly minimum variance unbiased estimator
MSE	Mean square error

Notations

$F_1(.)$	Cumulative distribution function of X
$F_2(.)$	Cumulative distribution function of Y
$\hat{R}_1$	MLE of R
$\hat{R}_2$	UMVUE of R
$\hat{R}_3$	Bayes estimator of R using conjugate prior
$\hat{R}_4$	Bayes estimator of R using non informative prior
$E(\hat{R}_1)$	Expected value of $\hat{R}_1$
$Var(\hat{R}_1)$	Variance of $\hat{R}_1$
$WE(\theta, k)$	Weibull distribution with parameters θ and k
$IG(a,\ b)$	Inverse gamma distribution with parameters a and b
$\chi^2(r)$	Chi-square distribution with parameters r
R	$P(Y\ < X)$
$\Gamma(.)$	Gamma function

1.Introduction

A common problem of interest in reliability analysis is that of estimating the probability that one variable exceeds another, that is,$R\ =\ P(Y < X)$, where $X\ and\ Y$ are independent random variables. The parameters, R is referred to the reliability parameter. This problem arises in the classical stress-strength reliability where one is interested in assessing the proportion of the times the random strength X of a component exceeds the random stress Y to which the component is subjected. This problem also arises in situation where $X\ and\ Y$ represent lifetimes of two devices and one wants to estimate the probability that one fails before the other.

Birnbaum (1956) was the first to consider the model $R\ =\ P(Y\ <\ X)$ and since then has found increasing number of applications in many different areas. If X is the strength of a system which is subjected to a stress Y, then R is a measure of system performance, the system fails if at any time the applied stress is greater than its strength.

The estimation of R is very common in the statistical literature. For example, Church and Harris (1970), Downton (1973), Tong (1974, 1977), Beg and Singh (1979), Awad, et.al.(1981), Sathe and Shah (1981), Johnson (1988), McCool (1991), Ivshin and Lumelskii (1995), Mahmoud (1996), Ahmed et.al.(1997), ,Surles and Padgett (2001), Abd-Elfattah and Mandouh (2004), Kundu and Gupta (2005, 2006). Recently, Kotz et al. (2003), presented a review of all methods and results on the stress-strength model in the last four decades.

Weibull is one of the most widely used distributions in reliability studies. It is often used as the lifetime distribution, because some failure models are described by their shape parameter. Therefore, the Weibull distribution is important and has been studied extensively over the years.

Censoring is very common in life tests. The most common censoring schemes are type I and type II. In many applications, test units may have to remove during test although they have not yet failed completely. Under censoring of type II, a random sample of n units is followed as long as necessary until r units have experienced the event. In this design the number of failures r, which determines the precision of yhe study, is fixed in advance and can be used a design parameter.

In this paper, we consider the problem of estimating reliability in the stress strength model when the strength of a unit or a system, X , has cumulative distribution function $F_1(x)$and the stress subject to it, Y, has CDF $F_2(y)$. The main purpose of this paper is to focus on the inference on$R = P(Y < X)$, where , $X$and Y are independent Weibull random variables with different scale parameters θ_1 *and* $\theta_2$respectively and common shape parameter k when the data are type II censored. The maximum likelihood estimator, uniformly minimum variance unbiased estimator and Bayes estimators of $P(Y < X)$are discussed. The maximum likelihood estimator and its asymptotic distribution are used to construct an asymptotic confidence interval of$P(Y < X)$.

We use the following notation. Weibull distribution with the scale parameter θ and shape parameter $k$will be denoted by $WE(\theta, k)$; and the corresponding density function is as follows $f(x, \theta, k) =\leq \frac{k}{\theta}\, x^{k-1}\, e^{-\frac{x^k}{\theta}},\ \ x > 0$

Moreover, the gamma density function with the shape and scale parameters a *and* $b$respectively will be denoted by $GA(a, b)$ and the corresponding density function is as follows$f(x, a, b) = \frac{b^a}{\Gamma a}\, x^{a-1}\, e^{-bx},\ \ x > 0$ where $\Gamma\,(\cdot)$is the gamma function. If Xfollows $GA(a, b)$then $\frac{1}{X}$ follows the inverse gamma, and it will be denoted by$IG(a, b)$

The rest of the paper is organized as follows. In Section 2, we obtain MLE of R and study some its properties. In Section 3, UMVUE of $R$is obtained. Bayesian estimators are presented in Section 4. Numerical illustrations have been used to compare different estimators in Section 5 using simulation study.

2. Maximum Likelihood Estimator of $R$with Type II Censored Samples

The MLE of R under Weibull distribution assumption has discussed by McCool (1991) in the complete sample case. To obtain the MLE of R based on type II censored sample, suppose X *and* Yfollow $WE(\theta_1, k)$ and $WE(\theta_2, k)$respectively, and they are independent. The probability density functions of $X$and $Y$are,

$$f(x, \theta_1, k) = \frac{k}{\theta_1} x^{k-1} e^{-\frac{x^k}{\theta_1}}, \quad x > 0 \tag{2.1}$$

$$f(y, \theta_2, k) = \frac{k}{\theta_2} y^{k-1} e^{-\frac{y^k}{\theta_2}}, \quad y > 0 \tag{2.2}$$

The reliability function is defined as

$$
\begin{aligned}
R = P(Y<X) &= \int_0^{\infty}\int_0^{x} f(y)\,f(x)\,dy\,dx \\
&= \int_0^{\infty} \tfrac{k}{\theta_1} x^{k-1}(1-e^{\frac{-x^k}{\theta_2}})e^{\frac{-x^k}{\theta_1}}\,dx \\
&= \tfrac{\theta_1}{\theta_1+\theta_2}
\end{aligned} \tag{2.3}
$$

Now to compute the MLE of$R$, first we need to obtain the MLE of θ_1 and θ_2. Suppose $X_1, ..., X_r$be a random sampled from Weibull distribution with parameters (θ_1, k) where $r \leq n$ are the first failure observations. The exact likelihood function with type-II censored sample is

$$
L(\theta_1, k) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(x_i)\,[1-F(x_i)]^{n-r}
$$

then

$$
L(\theta_1, k) = \frac{n!}{(n-r)!}\frac{k^r}{\theta_1^r}\prod_{i=1}^{r} x_i^{k-1}\, e^{\frac{-(\sum\limits_{i=1}^{r} x_r^k + (n-r)\; x_r^k)}{\theta_1}} \tag{2.4}
$$

$$
\frac{\partial \ln L(\theta_1, k)}{\partial \theta_1} = \frac{-r\hat{\theta}_1 + \sum\limits_{i=1}^{r} x_i^k + x_r^k(n-r)}{\hat{\theta}_1^2} = 0
$$

then

$$
\hat{\theta}_1 = \frac{\sum\limits_{i=1}^{r} x_i^k + x_r^k(n-r)}{r} \tag{2.5}
$$

Similarly,$Y_1, ..., Y_s$be a random sample from Weibull distribution with parameters (θ_2, k) where $s \leq m$ are the first failure observations.

$$
L(\theta_2, k) = \frac{m!}{(m-s)!}\frac{k^s}{\theta_2^s}\prod_{j=1}^{s} y_j^{k-1}\, e^{\frac{-(\sum\limits_{j=1}^{s} y_j^k + (m-s) y_s^k)}{\theta_2}} \tag{2.6}
$$

$$
\frac{\partial \ln L(\theta_2, k)}{\partial \theta_2} = \frac{-s\hat{\theta}_2 + \sum\limits_{j=1}^{s} y_j^k + y_s^k(m-s)}{\hat{\theta}_2^2} = 0
$$

then

$$
\hat{\theta}_2 = \frac{\sum\limits_{j=1}^{s} y_j^k + y_s^k(m-s)}{s} \tag{2.7}
$$

Once we obtain $\hat{\theta}_1$ and $\hat{\theta}_2$, the MLE of Rbecomes

$$\hat{R}_1 = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2} \tag{2.8}$$

from (2.5) and (2.7) in (2.8)

$$\hat{R}_1 = \frac{1}{1 + \frac{r}{s}\frac{\sum_{j=1}^{s} y_j^k + y_s^k (m-s)}{\sum_{i=1}^{r} x_i^k + x_r^k (n-r)}} \tag{2.9}$$

Since k is known, we have

$$\frac{2(\sum_{i=1}^{r} x_i^k + (n-r)x_r^k)}{\theta_1} \sim \chi^2(2(r+1))$$

and

$$\frac{2(\sum_{j=1}^{s} y_j^k + (m-s)y_s^k)}{\theta_2} \sim \chi^2(2(s+1))$$

Then $F = \frac{\hat{\theta}_1}{\hat{\theta}_2}(\frac{1}{\hat{R}_1} - 1)$ has F-distribution with $(2(s+1), 2\,(r+1))$degrees of freedom. From this fact we shall study some prosperities of $\hat{R}_1$. We can show that,

$$E\left(\hat{R}_1\right) = \frac{\theta_1}{\theta_1 + \frac{\theta_2 r}{(r-1)}}\left[1 - \frac{(r+s-1)}{s\,(r-2)}\left(1 - \frac{\theta_1}{\theta_1 + \frac{\theta_2 r}{(r-1)}}\right)^2\right] \tag{2.10}$$

For fixed s,

$$\lim_{r\to\infty} E\left(\hat{R}_1\right) = R\left[1 - \frac{1}{s}\,(1-R)^2\right] \tag{2.11}$$

and

$$\lim_{r,s\to\infty} E\left(\hat{R}_1\right) = R \tag{2.12}$$

from (2.12), $\hat{R}_1$ asymptotically unbiased estimator of R.
Also,

$$Var\left(\hat{R}_1\right) = \frac{(r+s-1)}{s\,(r-2)}\left[\frac{\frac{\theta_2 r}{\theta_1(s-1)}}{\frac{\theta_1(r+1)}{\theta_2(s-1)}}\right]^2\left[\frac{1}{1+\frac{\theta_2 r}{\theta_1(r-1)}}\right]^2 \tag{2.13}$$

$$\lim_{r,s\to\infty} Var\left(\hat{R}_1\right) = R^2\left[\frac{\theta_2}{\theta_1}\right]^4 \lim_{s\to\infty}\frac{1}{s} \tag{2.14}$$

then

$$\lim_{r,s\to\infty} Var\left(\hat{R}_1\right) = 0 \tag{2.15}$$

From (2.12) and (2.15), $\hat{R}_1$ is a consistent estimator for R.

3. Uniform Minimum Variance unbiased Estimator of R

Set $u_i = x_i^k$,$i = 1, ..., r$. Then $U = \sum\limits_{i=1}^{r} u_i$is minimal sufficient statistic for θ_1. Similarly, $v_j = y_j^k$,$j = 1, ..., s$. Then $V = \sum\limits_{j=1}^{s} v_j$ be minimal sufficient statistic for θ_2. Moreover (U, V) is minimal set of jointly complete and sufficient statistics for θ_1,θ_2.
Let

$$W = \begin{cases} 1, & v_1 < u_1 \\ 0, & v_1 \geq u_1 \end{cases}$$

$$E(W) = 1.P(v_1 < u_1) + 0.P(v_1 \geq u_1) = P(y_1^k < x_1^k) = P(y_1 < x_1) = R$$

Therefore $W$is an unbiased estimator for R. Then the UMVUE, $\hat{R}_2$for R is given by,

$$\begin{aligned}\hat{R}_2 &= E(W|\sum_{i=1}^{r} u_i, \sum_{j=1}^{s} v_j) \\ &= P(y_1 < x_1|\sum_{i=1}^{r} u_i, \sum_{j=1}^{s} v_j)\end{aligned}$$

By using Rao-Blackwell and Lehmann - Scheffe' Theorem to find UMVUE for$R$.(see Mood et al.(1974)).

$$\hat{R}_2 = \int\limits_{z_1}\int\limits_{v_1} w\, f(u_1, v_1\,|\,U, V)dv_1\,du_1$$

$u_1, v_1$are independent, we have

$$\hat{R}_2 = \int\limits_{z_1}\int\limits_{v_1} w\, f(u_1 \,|\, u) f(v_1 \,|\, v) dv_1\, du_1 \tag{3.1}$$

$$f(u_1 \,|\, u) = \frac{f(u_1) f(u - u_1)}{f(u)} \tag{3.2}$$

and

$$f(v_1 \,|\, v) = \frac{f(v_1) f(v - v_1)}{f(v)} \tag{3.3}$$

Note that U and V are independent gamma random variables with parameters (r, θ_1) and (s, θ_2), respectively.
We see that $U - u_1$ and $V - v_1$are independent gamma random with parameters $(r-1, \theta_1)$ and $(s-1,\, \theta_2)$, respectively. Moreover $U - u_1$ and $u_1$are independent, as well as $V - v_1$ and v_1 are also independent. We see that

$$\hat{R}_2 = \int_{u_1}\int_{v_1} w \frac{(r-1)(s-1)}{u^{(r-1)} v^{(s-1)}} (u - u_1)^{(r-2)} (v - v_1)^{(s-2)} dv_1 du_1, \tag{3.4}$$

Put $A = \dfrac{(r-1)(s-1)}{u^{(r-1)} v^{(s-1)}}$

$$\hat{R}_2 = A \begin{cases} \int_0^v (v - v_1)^{(r-2)} \dfrac{(u - v_1)^{(r-1)}}{r-1} dv_1, & v < u, \\ \int_0^z [\dfrac{v^{s-1}}{s-1} - \dfrac{(v - u_1)^{(s-2)}}{s-1}] (u - u_1)^{(s-2)} du_1, & v \geq u; \end{cases}$$

By using Binomial expansion, we have

$$\hat{R}_2 = \begin{cases} (r-1)!(s-1)! \sum\limits_{j=1}^{r-1} \dfrac{(-1)^j (\frac{v}{u})^j}{(r-1-j)!(s-1+j)!}, & v < u, \\ 1 - (r-1)!(s-1)! \sum\limits_{j=1}^{r-1} \dfrac{(-1)^j (\frac{u}{v})^j}{(s-1-j)!(r-1+j)!}, & v \geq u; \end{cases} \tag{3.5}$$

where

$$U = \sum_{i=1}^{r} u_i \text{ and } V = \sum_{j=1}^{s} v_j. \tag{3.6}$$

4. Bayes Estimator

In this section, we consider Bayesian inference on R. We obtain Bayes estimate of R under the square error loss function based on cojuagate and noninformative priors of the parameters θ_1 and θ_2.

4.1 Conjugate prior distribution

Let $X_1, ..., X_r$ and $Y_1, ..., Y_s$ be the first r and s failure observations from $X_1, ...,$ X_n and $Y_1, ..., Y_m$ respectively. Both of them have Weibull distribution with parameters (θ_1, k) and (θ_2, k) respectively. According to approach of Berger and Sun (1993), it is assumed that the prior density of θ_1 is inverted $IG(a, b)$, therefore the prior density function of θ_1 becomes we will choice the prior distribution of θ_1 is given by

$$\text{(4.1)} \qquad \pi_{01}(\theta_1) = \frac{b^a e^{-\frac{b}{\theta_1}} \theta_1^{(a+1)}}{\Gamma(a)}, \quad \theta_1 > 0$$

The joint of the likelihood function with type II censored sample is:

$$\text{(4.2)} \qquad f\left(x_1, ..., x_r \left| \theta_1 \right.\right) = \frac{n!}{(n-r)!} \frac{k^r}{\theta_1^r} \prod_{i=1}^{r} x_i^{k-1} e^{\frac{-(\sum\limits_{i=1}^{r} x_i^k + (n-r) x_r^k)}{\theta_1}}$$

then the posterior function of θ_1

$$\text{(4.3)} \qquad \pi_1(\theta_1) = f\left(\theta_1 \left| x_1, ..., x_r \right.\right) = \frac{e^{\frac{-\lambda_1}{\theta_1}} \lambda_1^{1+r+b}}{\theta_1^{(r+b)} \Gamma(r+b+1)}$$

where$\lambda_1 = a + \sum\limits_{i=1}^{r} x_i^k + (n-r) x_r^k$.
Similarly, let the prior of θ_2

$$\text{(4.4)} \qquad \pi_{02}(\theta_2) = \frac{c^d e^{-\frac{c}{\theta_2}}}{\Gamma(d)} \theta_2^{-(d+1)}, \theta_2 > 0$$

then the posterior function of θ_2

$$\text{(4.5)} \qquad \pi_2(\theta_2) = f\left(\theta_2 \left| y_1, ..., y_s \right.\right) = \frac{\lambda_2^{d+s+1} e^{\frac{-\lambda_2}{\theta 2}}}{\Gamma(s+d+1) \theta_2^{(s+d)}}$$

Where $\lambda_2 = c + \sum\limits_{j=1}^{s} y_j^k + (m-s) y_s^k$

Since both θ_1 and θ_2 are independent then the joint posterior distribution function is

$$\pi\left(\theta_1, \theta_2 \left| x_1, ..., x_r\right. ; y_1, ..., y_s\right) = \frac{\lambda_1^{b+r+1} \lambda_2^{d+s+1} e^{\frac{-\lambda_1}{\theta_1} - \frac{\lambda_2}{\theta_2}}}{\Gamma(r+b+1)\Gamma(s+d+1)\theta_1^{(r+b)}\theta_2^{(s+d)}} \tag{4.6}$$

Hence Bayes estimator of R with respect to the mean square error loss function is

$$\hat{R}_3 = E\left(R \left| x_1, ..., x_r\right. ; y_1, ..., y_s\right)$$

then

$$\hat{R}_3 = \frac{\lambda_1^{b+r+1}\lambda_2^{d+s+1}}{\Gamma(r+b+1)\Gamma(s+d+1)} \int_0^{\infty} \frac{\Gamma(r+s+b+d+1) R^{s+d+2} (1-R)^{r+b+1}}{(\lambda_1(1-R)+\lambda_2 R)^{r+s+b+d+1}} dR \tag{4.7}$$

4.2 Non Informative Prior Distributions

Let $X_1, ..., X_r$ be a random sample from Weibull distribution with parameters (θ_1, k). The prior distribution of θ_1 is proportional to $\sqrt{I(\theta_1)}$, where $I(\theta_1)$ is Fisher's information of the sample about θ_1, and is given by

$$I(\theta_1) = \frac{1}{\theta_1^2} + 2\frac{\Gamma^k(1+\frac{1}{k})}{\theta_1^3} \tag{4.8}$$

from that the Jeffrey's prior distribution

$$\pi_3 \propto \frac{1}{\theta_1} \tag{4.9}$$

Similarly, if $Y_1, ..., Y_s$ is a random sample from Weibull distribution with parameters (θ_2, k), the prior distribution of θ_2 will be given by:

$$\pi_4 \propto \frac{1}{\theta_2} \tag{4.10}$$

if we have θ_1 and θ_2 are independent then the posterior joint distribution of θ_1 and θ_2,will be

(4.11)

$$\pi\left(\theta_1, \theta_2 | x_1, ..., x_r, y_1, ..., y_s\right) \propto L\left(x_1, ..., x_r \left| \theta_1\right.\right) L\left(y_1, ..., y_s \left| \theta_2\right.\right) \pi_1(\theta_1)\pi_2(\theta_2)$$

Let

$$H_1 = \sum_{i=1}^{r} x_i^{k-1} + x_r^k(n-r) \text{ and } H_2 = \sum_{j=1}^{s} y_j^{k-1} + y_s^k(m-s)$$

then

$$\pi(\theta_1, \theta_2 | x, y) = \frac{H_1^r H_2^s}{\theta_1^{r+1}\theta_2^{s+1}\Gamma(r)\Gamma(s)} e^{\frac{-H_1}{\theta_1}} e^{\frac{-H_2}{\theta_2}}, \theta_1, \theta_2 > 0 \tag{4.12}$$

Under the mean square error, Bayes estimator $\hat{R}_4$ of R will be

$$\hat{R}_4 = E\left(R\,|x, y\right) = \frac{H_1^r H_2^s}{\Gamma(r)\Gamma(s)} \int_0^1 \frac{\Gamma(r+s+3)R^{s+3}(1-R)^{r+2}}{(H_1(1-R)+H_2R)^{r+s+3}}\, dR \tag{4.13}$$

5. Simulation study for the different estimators

In this section, we perform some simulation experiments to observe the baehavior of the different methods for different sample sizes and for different parameter values. We used the software package MathCad 2001 fo this purpose. We compare, in terms of the mean square error, the performances of the MLE, UMVUE and Bayes estimates with respect to squares error loss function. The following steps will be considered to obtain the estimators:

Step (1): Generate random samples $X_1, ..., X_r$ from Weibull distribution, we consider the following sample sizes (n, m) = (5,5), (10,10), (15,15), (20,20), (5,4), (10,5), (10,15), (10,20), (15,5), (15,10), (15,20), (20,10), and the following parameter values $\theta_1 = 2$, $\theta_2 = 3, 2$ and $k = 1.5$ with different type II censoring at 60%, 70%, 80% and 90% . We will generate 1000 random samples from Weibull distribution.

Step (2): Similarly, we generate samples for Weibull distribution, with parameters θ_2 and k.

Step (3): Using the Equation (2.8) to find the MLE of R and the Equation (3.5) to find the UMVUE of R. Also using the equation (4.7) the values of Bayes estimator of R is obtained using Conjugate prior distribution. Finally, the equation (4.13) gives the estimators of R using non informative prior distribution. The results are based on 1000 replications.

Step (4): We take the average of the simulated values and calculate the the mean square error of R. The results are reported in Tables (1) – (4).

From the tables, we find the following:

When the sample sizes n and m, increase then the average mean square error decrease as expected in all the estimation methods. It is observed that the UMVUE and Bayes behave almost in a similar manner both with respect to MSE. The MLE estimate behaves quite different from the other. It has significantly lower MSE in most of the cases.

Tables (1) ,(2)

1. We will find that MSE of $\hat{R}_1$has the smallest values among the other values of MSE of [($\hat{R}_2$), ($\hat{R}_3$) and ($\hat{R}_4$)] expect at some points $\hat{R}_2$has advantage over the other estimators.
2. At some points MSE $\hat{R}_3$ is better than MSE of $\hat{R}_4$.
3. All mean square errors decrease as $\theta_1$and θ_2 increases.

Tables (3) ,(4)

1. We will find that MSE of $\hat{R}_2$has the smallest values among the other values of MSE of [($\hat{R}_1$), ($\hat{R}_3$) and ($\hat{R}_4$)] expect at some points $\hat{R}_1$has advantage over the other estimators.
2. At some points MSE $\hat{R}_3$ is better than MSE of $\hat{R}_4$.
3. All mean square errors decrease when $r \neq s$.

n	m	r	s	R_1	MSE_1	R_2	MSE_2	R_3	MSE_3	R_4	MSE_4
5	5	3	3	0.435	1.211E-03	0.383	2.976E-04	0.086	9.900E-02	0.119	7.900E-02
5	5	4	4	0.423	5.344E-04	0.401	2.710E-07	0.155	6.000E-02	0.328	5.192E-03
10	10	6	6	0.456	3.189E-03	0.412	6.243E-03	0.141	6.700E-02	0.188	6.500E-02
10	10	7	7	0.494	8.898E-03	0.388	3.467E-04	0.204	3.900E-02	0.308	8.262E-03
10	10	8	8	0.424	5.773E-04	0.415	2.249E-04	0.289	1.200E-02	0.156	6.000E-02
10	10	9	9	0.464	6.040E-03	0.427	7.290E-04	0.300	1.000E-02	0.542	1.500E-02
15	15	12	12	0.449	2.392E-03	0.439	1.558E-03	0.191	6.400E-02	0.238	2.600E-02
15	15	13	13	0.432	5.323E-04	0.412	1.452E-04	0.331	6.734E-03	0.496	9.199E-03
20	20	15	15	0.433	1.063E-03	0.416	2.522E-04	0.145	6.500E-02	0.159	5.800E-02
20	20	16	16	0.401	2.069E-06	0.385	2.317E-04	0.222	3.200E-02	0.268	1.700E-02
20	20	17	17	0.459	3.435E-03	0.454	2.885E-03	0.372	7.720E-04	0.494	8.794E-03

Table (1) When $\theta_1 = 2$, $\theta_2 = 3$, $k = 1.5$ and $R = 0.4$

n	m	r	s	R_1	MSE_1	R_2	MSE_2	R_3	MSE_3	R_4	MSE_4
5	5	3	3	0.491	7.39E-05	0.482	3.18E-04	0.03	9.200E-02	0.016	1.010E-01
5	5	4	4	0.479	6.32E-04	0.47	9.087E-04	0.016	1.550E-01	0.137	1.320E-01
10	10	6	6	0.556	3.09E-03	0.583	6.872E-03	0.035	2.170E-01	0.023	2.270E-01
10	10	7	7	0.561	3.76E-03	0.589	7.944E-03	0.073	1.820E-01	0.069	1.860E-01
10	10	8	8	0.499	1.39E-06	0.499	2.241E-06	0.151	1.220E-01	0.19	9.600E-02
10	10	9	9	0.469	9.60E-04	0.465	1.231E-03	0.305	3.800E-02	0.54	1.562E-03
15	15	12	12	0.531	9.76E-04	0.537	1.375E-03	0.227	7.500E-02	0.186	9.900E-02
15	15	13	13	0.554	2.89E-03	0.561	3.684E-03	0.347	2.300E-02	0.534	1.151E-03
20	20	15	15	0.561	2.46E-04	0.519	3.50E-04	0.141	1.290E-01	0.152	1.210E-01
20	20	16	16	0.566	6.31E-03	0.575	5.566E-03	0.218	7.900E-02	0.263	5.600E-02
20	20	17	17	0.499	8.21E-07	0.499	1.009E-06	0.37	1.700E-02	0.514	1.906E-04

Table (2) When $\theta_1 = 2$, $\theta_2 = 2$, $k = 1.5$ and $R = 0.5$

n	m	r	s	R_1	MSE_1	R_2	MSE_2	R_3	MSE_3	R_4	MSE_4
5	4	3	2	0.433	1.107E-03	0.414	2.033E-04	0.017	1.470E-01	0.01	1.520E-01
5	5	4	3	0.448	2.270E-03	0.369	9.746E-04	0.04	1.300E-01	0.047	1.240E-01
10	5	6	5	0.468	6.629E-03	0.479	6.276E-03	0.107	8.600E-02	0.147	6.400E-02
10	10	7	6	0.396	1.600E-05	0.385	1.659E-03	0.044	1.260E-01	0.031	1.360E-01
10	15	8	7	0.403	8.140E-06	0.395	3.01E-05	0.027	1.390E-01	0.015	1.480E-01
10	20	9	10	0.401	7.196E-08	0.42	6.053E-04	0.034	1.340E-01	0.022	1.430E-01
15	5	6	5	0.422	6.840E-04	0.401	5.379E-07	0.037	1.320E-01	0.035	1.330E-01
15	10	7	6	0.431	9.909E-04	0.389	1.210E-04	0.389	1.210E-04	0.389	1.210E-04
15	15	8	7	0.408	6.520E-05	0.389	1.210E-04	0.389	1.210E-04	0.389	1.210E-04
15	20	10	10	0.404	1.758E-05	0.442	1.729E-03	0.027	1.390E-01	0.018	1.460E-01
20	10	16	9	0.448	2.334E-03	0.415	2.346E-04	0.329	5.077E-03	0.401	5.787E-06
20	15	17	14	0.464	6.113E-03	0.446	2.091E-03	0.366	1.156E-03	0.355	2.025E-03

Table (3) When $\theta_1 = 2$, $\theta_2 = 3$, $k = 1.5$ and $R = 0.4$

n	m	r	s	R_1	MSE_1	R_2	MSE_2	R_3	MSE_3	R_4	MSE_4
5	4	3	2	0.458	1.762E-03	0.458	3.409E-03	0.037	1.370E-01	0.355	2.025E-03
5	5	4	3	0.45	2.471E-03	0.55	2.468E-03	0.056	1.970E-01	0.355	2.100E-02
10	5	6	5	0.592	8.760E-03	0.588	7.753E-03	0.115	1.480E-01	0.1722	1.080E-01
10	10	7	6	0.498	3.964E-06	0.567	6.475E-03	0.053	2.000E-01	0.039	2.130E-01
10	15	8	7	0.473	7.161E-04	0.517	5.180E-04	0.03	2.210E-01	0.016	2.340E-01
10	20	9	10	0.505	2.399E-05	0.52	6.080E-04	0.035	2.160E-01	0.022	2.280E-01
15	5	6	5	0.521	6.410E-04	0.422	6.093E-03	0.037	2.140E-01	0.035	2.170E-01
15	10	7	6	0.561	3.680E-03	0.513	1.672E-04	0.023	2.280E-01	0.015	2.360E-01
15	15	8	7	0.513	1.787E-04	0.557	3.301E-03	0.016	2.350E-01	0.035	2.114E-01
15	20	10	10	0.46	1.593E-03	0.493	5.358E-05	0.022	2.290E-01	0.015	2.350E-01
20	10	16	9	0.508	5.975E-05	0.489	1.147E-04	0.334	2.800E-02	0.495	2.452E-05
20	15	17	14	0.536	1.314E-03	0.525	6.245E-04	0.51	1.084E-04	0.399	1.000E-02

Table (4) When $\theta_1 = 2$, $\theta_2 = 2$, $k = 1.5$ and $R = 0.5$

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